Game Programming

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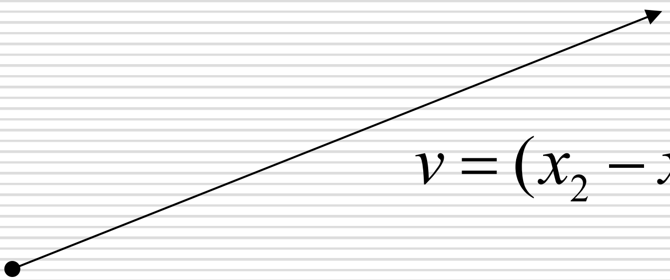
Game Mathematics

- Vectors
- Matrices
- Transformations
- Homogeneous Coordinates
- 3D Viewing
- Triangle Mathematics
- Intersection Issues
- Fixed-point Real Numbers
- Quaternions
- Parametric Curves

Vectors

- A vector is an entity that possesses *magnitude* and *direction*.
- A ray (directed line segment), that possesses *position*, *magnitude*, and *direction*.

(x_2, y_2, z_2)



$$v = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

(x_1, y_1, z_1)

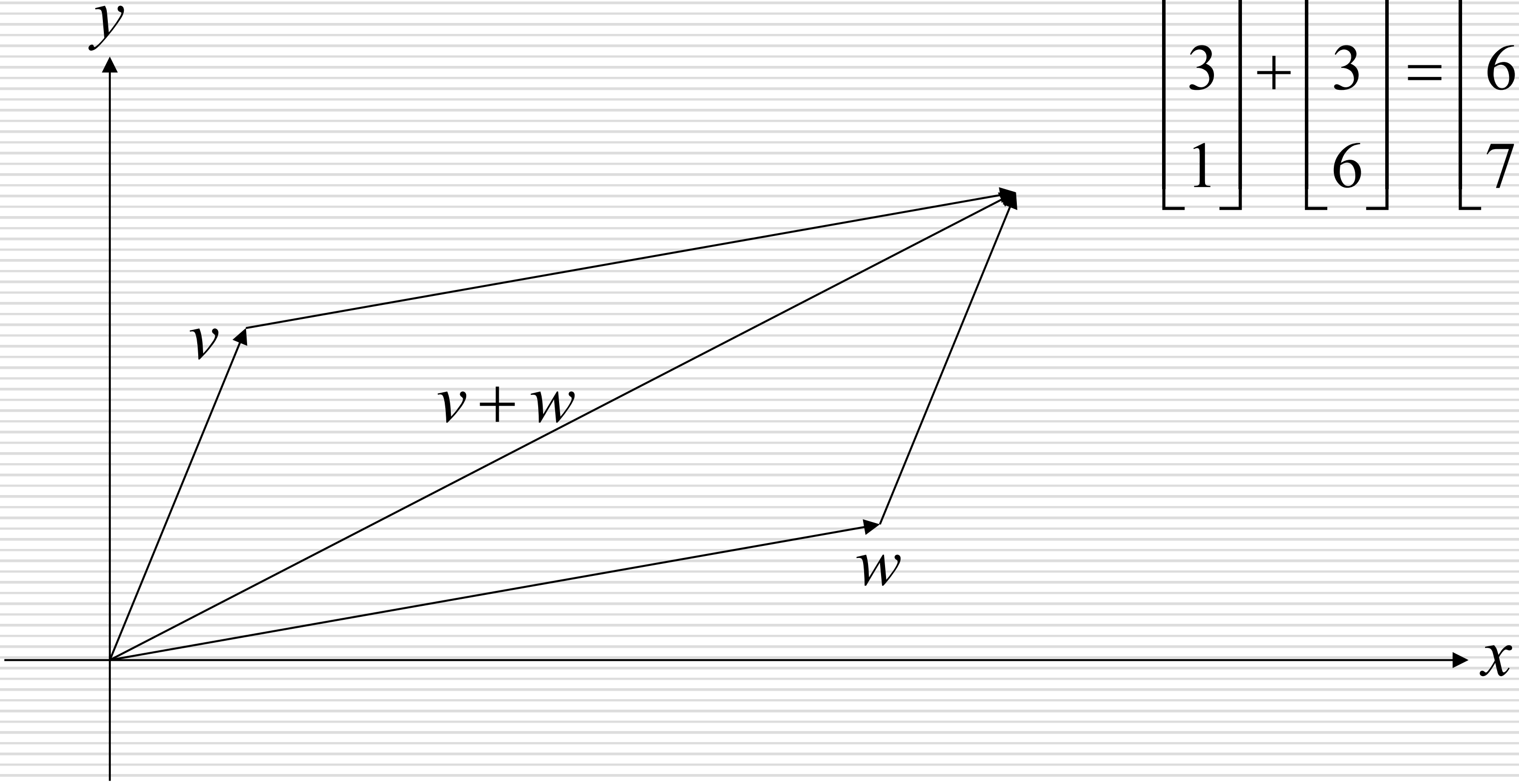
Vectors

- an n-tuple of real numbers (scalars)
- two operations: addition & multiplication
- Commutative Laws
  - $a + b = b + a$
  - $a \cdot b = b \cdot a$
- Identities
  - $a + 0 = a$
  - $a \cdot 1 = a$
- Associative Laws
  - $(a + b) + c = a + (b + c)$
  - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Distributive Laws
  - $a \cdot (b + c) = a \cdot b + a \cdot c$
  - $(a + b) \cdot c = a \cdot c + b \cdot c$
- Inverse
  - $a + b = 0 \rightarrow b = -a$

Addition of Vectors

- parallelogram rule

$$\begin{bmatrix}1\\3\\1\end{bmatrix}+\begin{bmatrix}2\\3\\6\end{bmatrix}=\begin{bmatrix}3\\6\\7\end{bmatrix}$$



The Vector Dot (Inner) Product

$$u = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

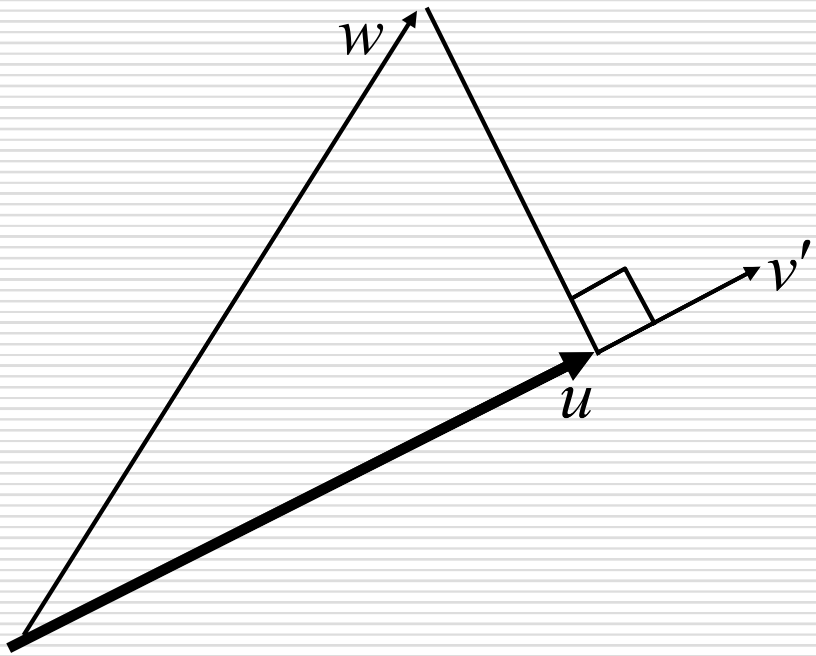
$$\Rightarrow u \bullet v = x_1 y_1 + \ldots + x_n y_n$$

- length = $\sqrt{u \bullet u} = \|u\|$

Properties of the Dot Product

- symmetric
  - $v \bullet w = w \bullet v$
- nondegenerate
  - $v \bullet v = 0$ only when $v = 0$
- bilinear
  - $v \bullet (u + \alpha w) = v \bullet u + \alpha (v \bullet w)$
- unit vector (normalizing)
  - $v' = v / \|v\|$
- angle between the vectors
  - $\cos^{-1}\left(v \bullet w / \|v\| \|w\|\right)$

Projection



$$\|u\| = \|w\|\cos\theta$$

$$= \|w\|\left(\frac{v' \bullet w}{\|v'\|\|w\|}\right)$$

$$= v' \bullet w$$

Cross Product of Vectors

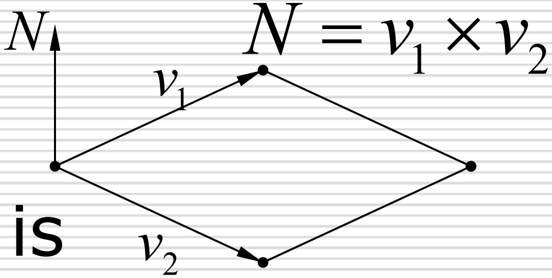
- Definition

$$x = v \times w$$

$$= (v_2 w_3 - v_3 w_2)i + (v_3 w_1 - v_1 w_3)i + (v_1 w_2 - v_2 w_1)k$$

  - where $i = (1,0,0), j = (0,1,0), k = (0,0,1)$ are standard unit vectors

- Application
  - A normal vector to a polygon is calculated from 3 (non-collinear) vertices of the polygon.



Vector in Unity

- Vector2
  - Representation of 2D vectors and points (e.g., texture coordinates in a Mesh or texture offsets in Material).
  - In the majority of other cases a Vector3 is used.
- Vector2Int
  - using integers.

Vector in Unity

- Vector3
  - Representation of 3D vectors and points.
  - this structure is used throughout Unity to pass 3D positions and directions around.
- Vector3Int
  - using integers.
- Vector4
  - Representation of four-dimensional vectors. (e.g., mesh tangents, parameters for shaders)

Vector3 in Unity

- Static Properties

back	Shorthand for writing Vector3(0, 0, -1).
down	Shorthand for writing Vector3(0, -1, 0).
forward	Shorthand for writing Vector3(0, 0, 1).
left	Shorthand for writing Vector3(-1, 0, 0).
one	Shorthand for writing Vector3(1, 1, 1).
right	Shorthand for writing Vector3(1, 0, 0).
up	Shorthand for writing Vector3(0, 1, 0).
zero	Shorthand for writing Vector3(0, 0, 0).

Vector3 in Unity

- Properties

magnitude	Returns the length of this vector (Read Only).
normalized	Returns this vector with a magnitude of 1 (Read Only).
sqrMagnitude	Returns the squared length of this vector (Read Only).
this[int]	Access the x, y, z components using [0], [1], [2] respectively.
x	X component of the vector.
y	Y component of the vector.
z	Z component of the vector.

Vector3 in Unity

- Static Methods

Angle	Returns the angle in degrees between from and to.
ClampMagnitude	Returns a copy of vector with its magnitude clamped to maxLength.
Cross	Cross Product of two vectors.
Distance	Returns the distance between a and b.
Dot	Dot Product of two vectors.
Lerp	Linearly interpolates between two vectors.
MoveTowards	Moves a point current in a straight line towards a target point.
Normalize	Makes this vector have a magnitude of 1.
OrthoNormalize	Makes vectors normalized and orthogonal to each other.

Vector3 in Unity

- Static Methods

Project	Projects a vector onto another vector.
ProjectOnPlane	Projects a vector onto a plane defined by a normal orthogonal to the plane.
Reflect	Reflects a vector off the plane defined by a normal.
RotateTowards	Rotates a vector current towards target.
Slerp	Spherically interpolates between two vectors.

Matrix Basics

- Definition

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

- Transpose

$$\mathbf{C} = \mathbf{A}^T \quad c_{ij} = a_{ji} \Rightarrow \mathbf{C} = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \dots & a_{nm} \end{bmatrix}$$

- Addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad c_{ij} = a_{ij} + b_{ij}$$

Matrix Basics

- Scalar-matrix multiplication

$$\mathbf{C} = \alpha\mathbf{A} \qquad c_{ij} = \alpha a_{ij}$$

- Matrix-matrix multiplication

$$\mathbf{C} = \mathbf{AB} \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Matrix multiplication are ***not commutative***

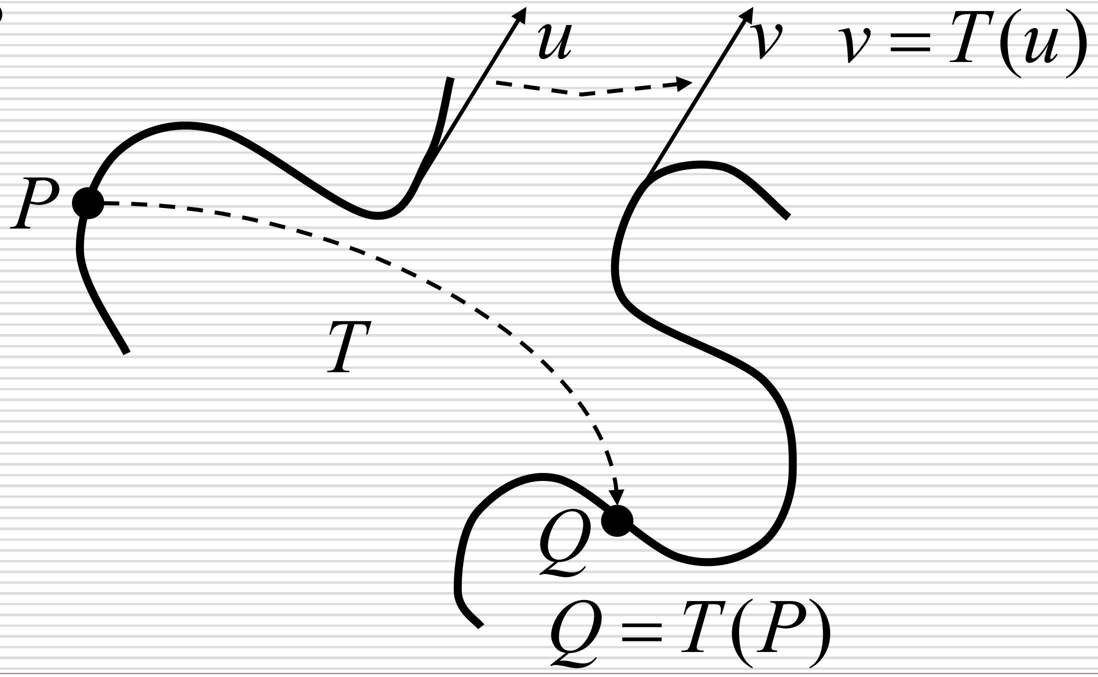
$$\mathbf{AB} \neq \mathbf{BA}$$

Matrix in Unity

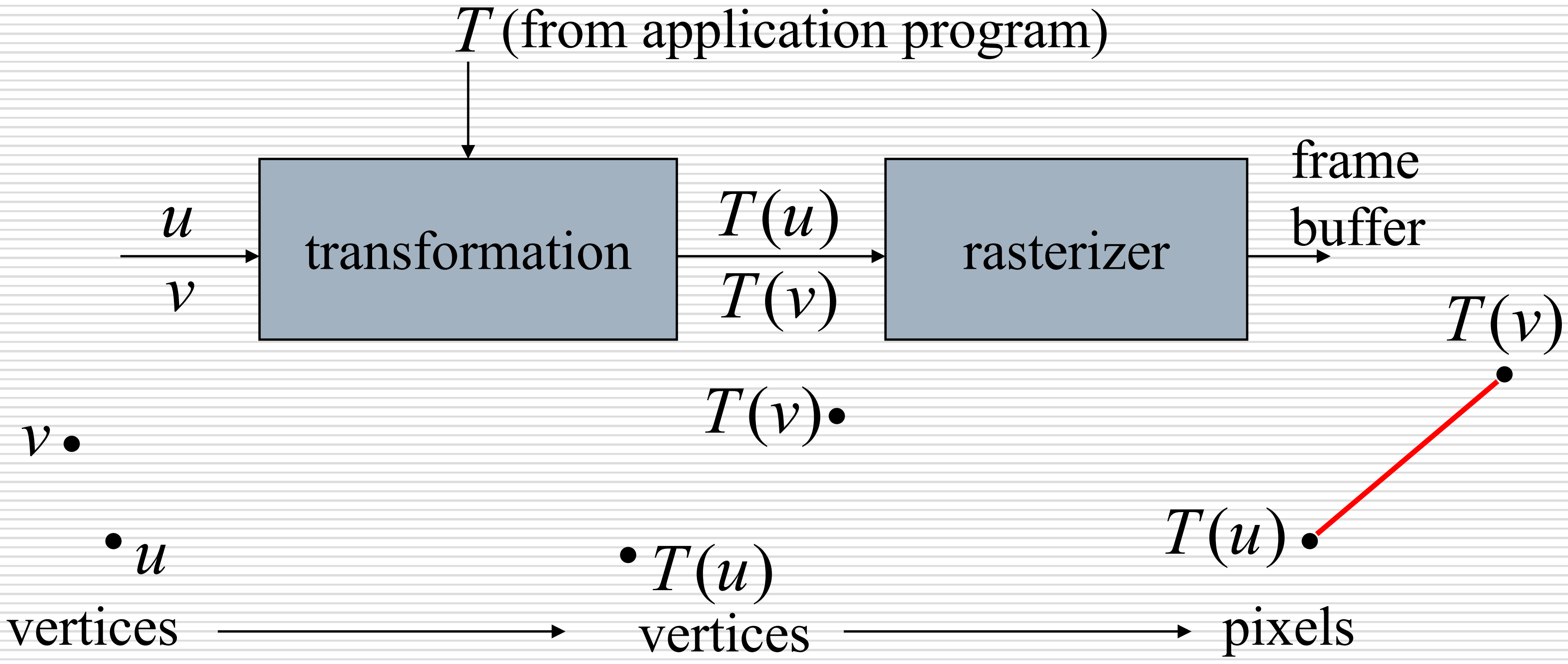
- Matrix4x4
  - You rarely use matrices in scripts; most often using Vector3s, Quaternions and functionality of Transform class is more straightforward.
  - Setting up nonstandard camera projection.
  - In Unity, Matrix4x4 is used by several Transform, Camera, Material and GL functions.
    - Transform.localToWorldMatrix
    - Transform.worldToLocalMatrix
    - Camera.projectionMatrix
    - Camera.worldToCameraMatrix
    - …

General Transformations

- A transformation maps points to other points and/or vectors to other vectors



Pipeline Implementation



Representation

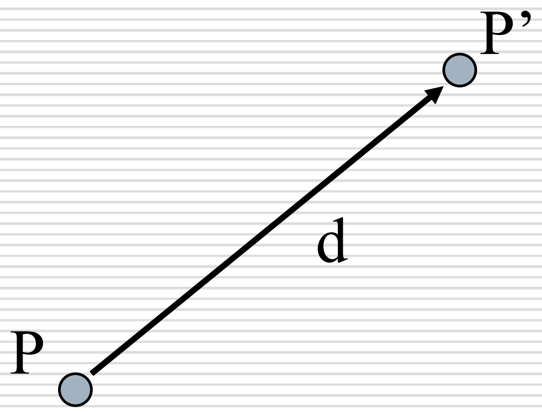
- We can represent a **point**, $\mathbf{p} = (x, y)$ in the plane
  - as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$
  - as a row vector $\begin{bmatrix} x & y \end{bmatrix}$

2D Transformations

- 2D Translation
- 2D Scaling
- 2D Reflection
- 2D Shearing
- 2D Rotation

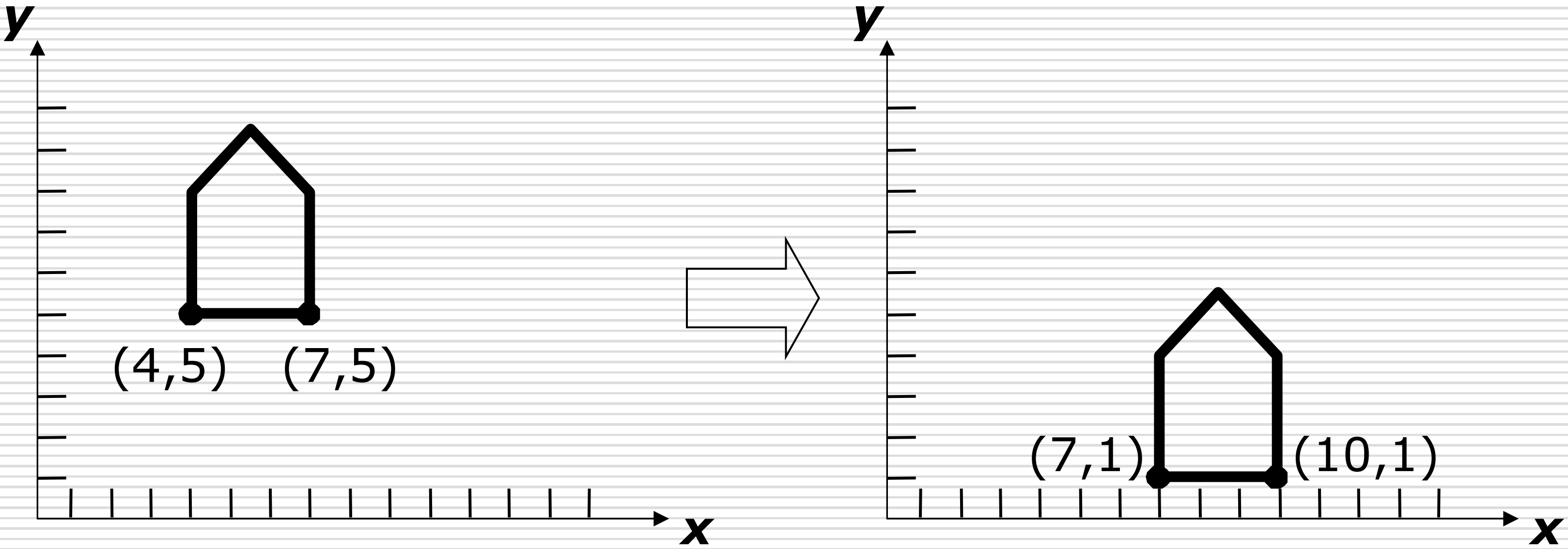
Translation

- Move (translate, displace) a point to a new location



- Displacement determined by a vector d
  - Three degrees of freedom
  - P'=P+d

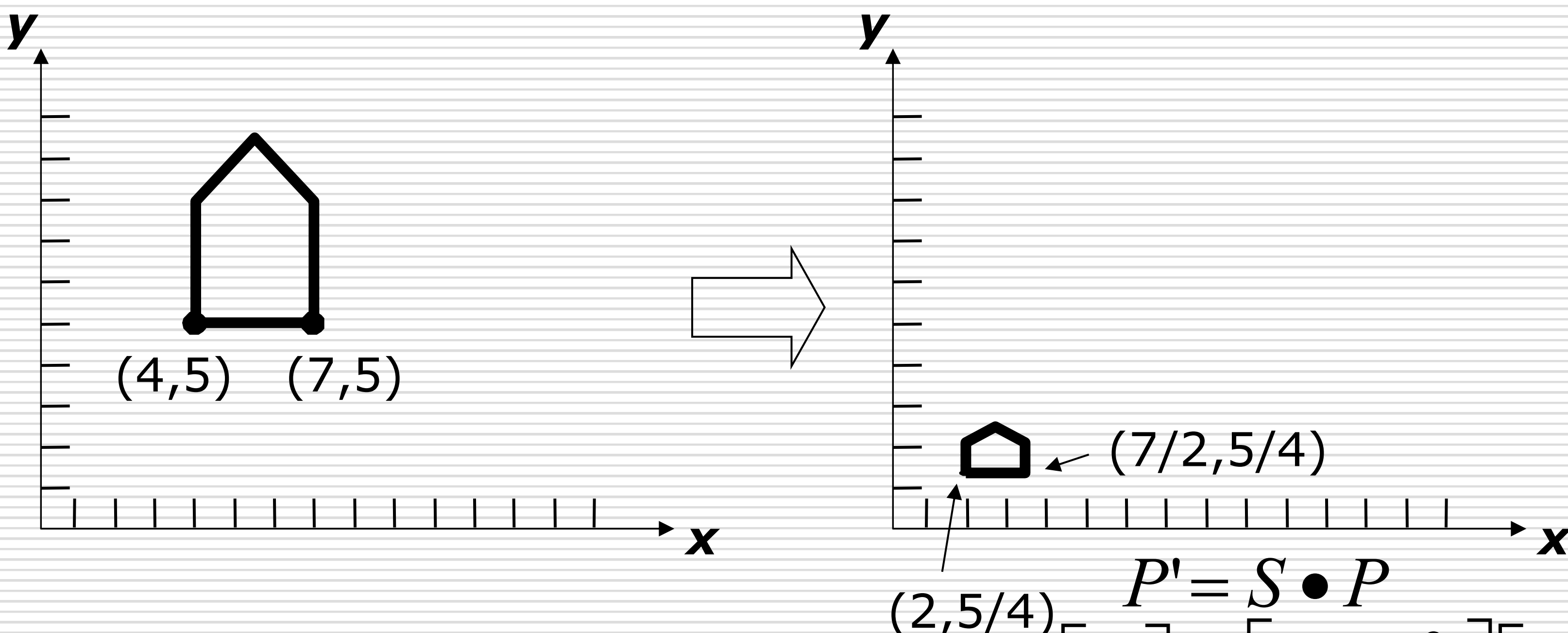
2D Translation



$$P' = P + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

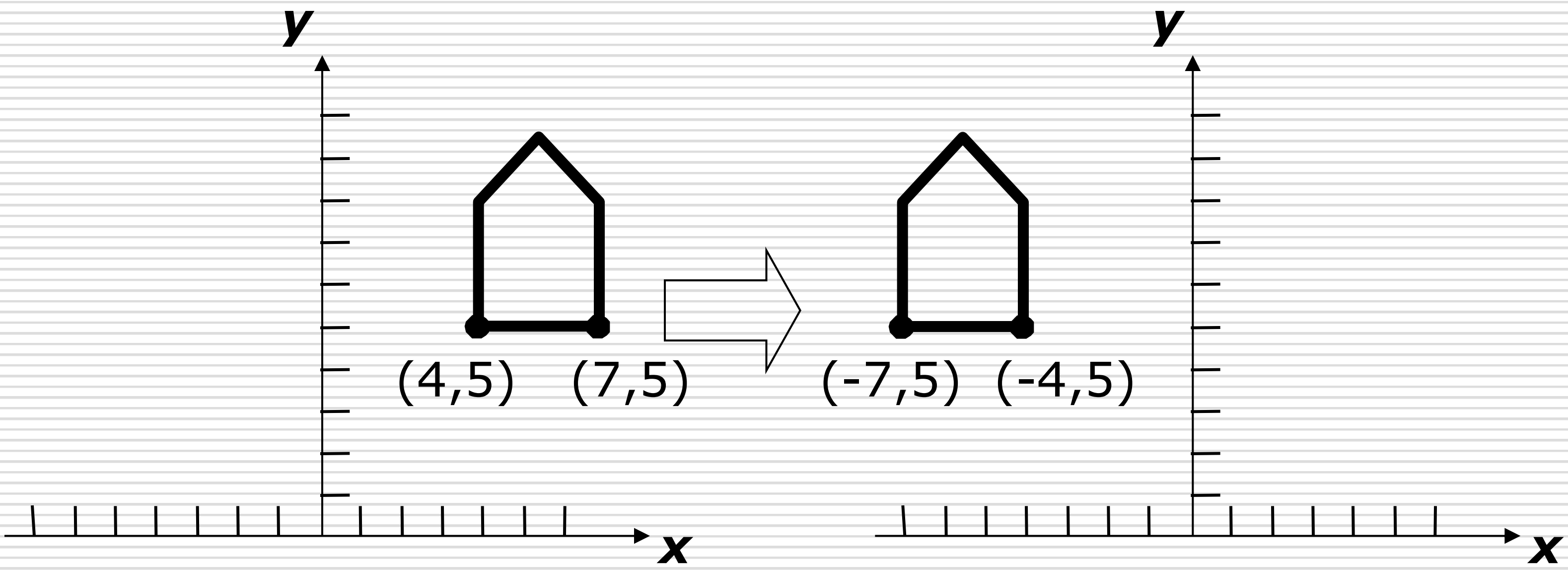
2D Scaling



$$P' = S \bullet P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

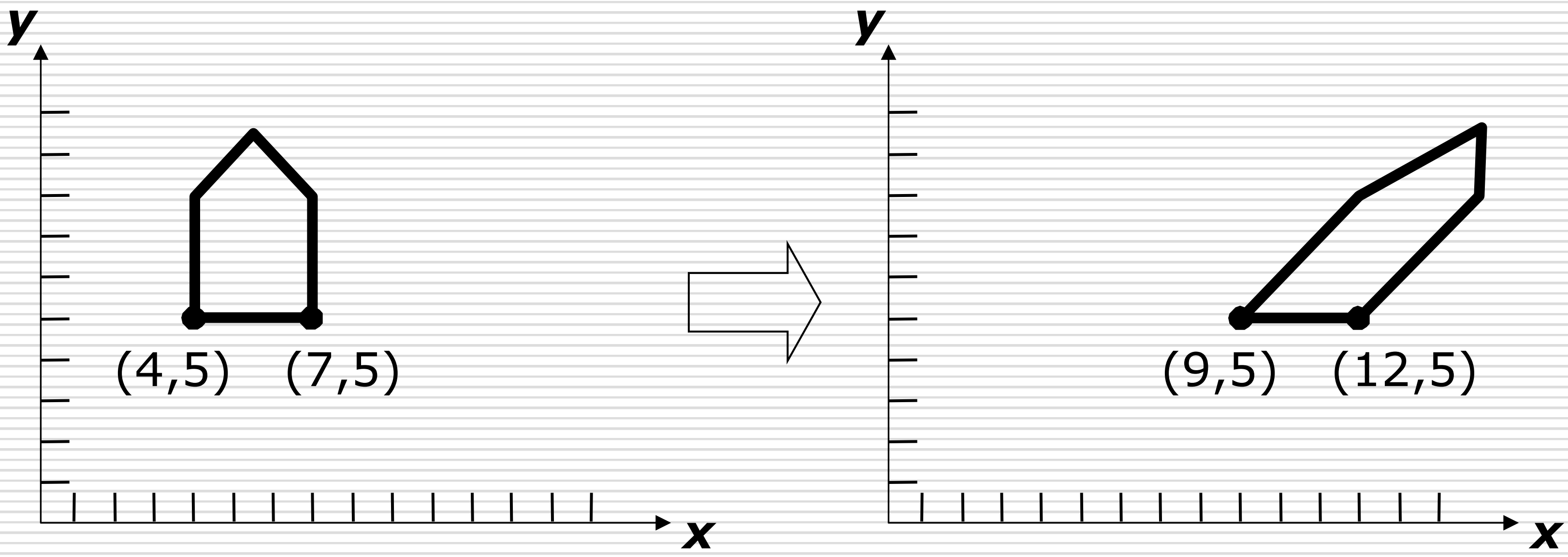
2D Reflection



$$P' = RE_x \bullet P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

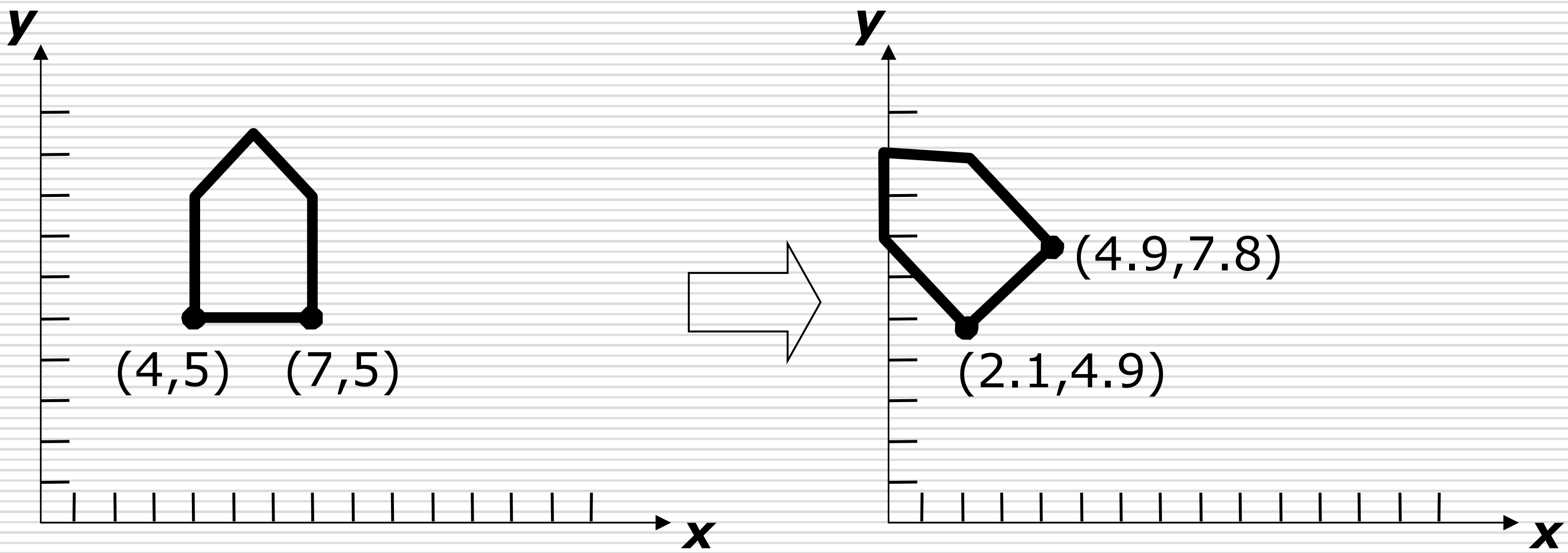
2D Shearing



$$P' = SH_x \bullet P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Rotation



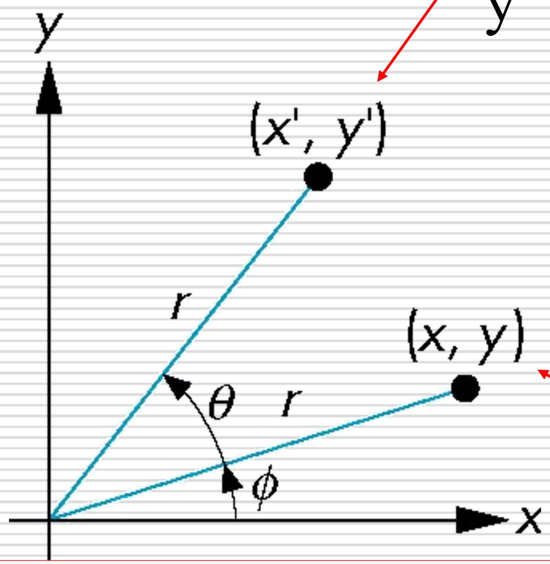
$$P' = R \bullet P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Rotation

- Consider rotation about the origin by θ degrees
  - radius stays the same, angle increases by θ

$$x' = r \cos(\phi + \theta)$$
$$y' = r \sin(\phi + \theta)$$



$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$

$$x = r \cos\phi$$
$$y = r \sin\phi$$

Limitations of a 2X2 matrix

- Scaling
- Rotation
- Reflection
- Shearing

- What do we miss?

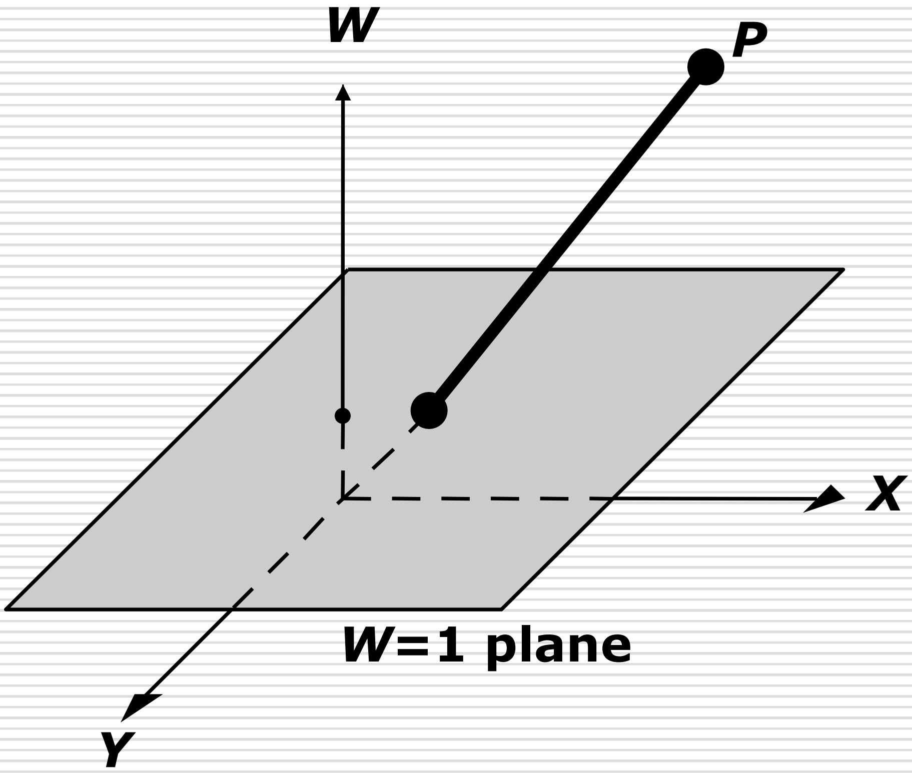
Homogeneous Coordinates

- Why & What is **homogeneous coordinates** ?
  - if points are expressed in homogeneous coordinates, all three transformations can be treated as multiplications.

$$(x, y) \rightarrow (x, y, W)$$

usually 1
can not be 0

Homogeneous Coordinates



Homogeneous Coordinates for 2D Translation

$$P' = P + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

$$P' = T(d_x, d_y) \bullet P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = T(d_{x1}, d_{y1}) \bullet P$$

$$P'' = T(d_{x2}, d_{y2}) \bullet P'$$

Homogeneous Coordinates for 2D Translation

$$P'' = T(d_{x2}, d_{y2}) \bullet (T(d_{x1}, d_{y1}) \bullet P)$$
$$= (T(d_{x2}, d_{y2}) \bullet T(d_{x1}, d_{y1})) \bullet P$$

$$T(d_{x2}, d_{y2}) \bullet T(d_{x1}, d_{y1}) = \begin{bmatrix} 1 & 0 & d_{x2} \\ 0 & 1 & d_{y2} \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & d_{x1} \\ 0 & 1 & d_{y1} \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & d_{x1} + d_{x2} \\ 0 & 1 & d_{y1} + d_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates for 2D Scaling

$$P' = S \bullet P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S(s_x, s_y) \bullet P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$S(s_{x2}, s_{y2}) \bullet S(s_{x1}, s_{y1}) = \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_{x1} \bullet s_{x2} & 0 & 0 \\ 0 & s_{y1} \bullet s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates for 2D Rotation

$$P' = R \bullet P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$$

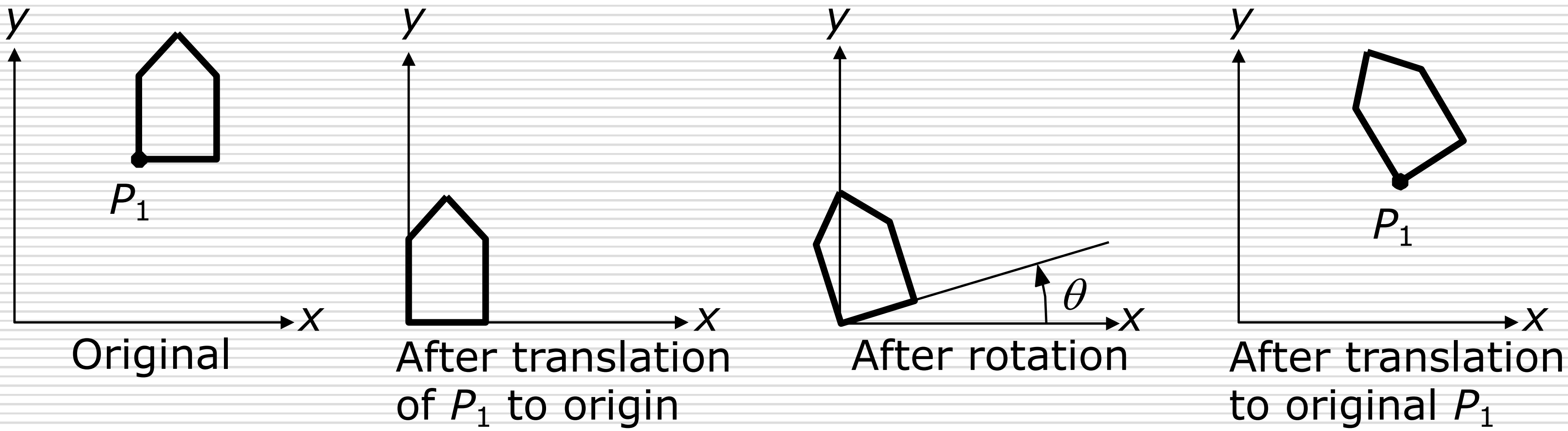
$$P' = R(\theta) \bullet P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Properties of Transformations

- rigid-body transformations
  - rotation & translation
  - preserving angles and lengths
- affine transformations
  - rotation & translation & scaling
  - preserving parallelism of lines

Composition of 2D Transformations



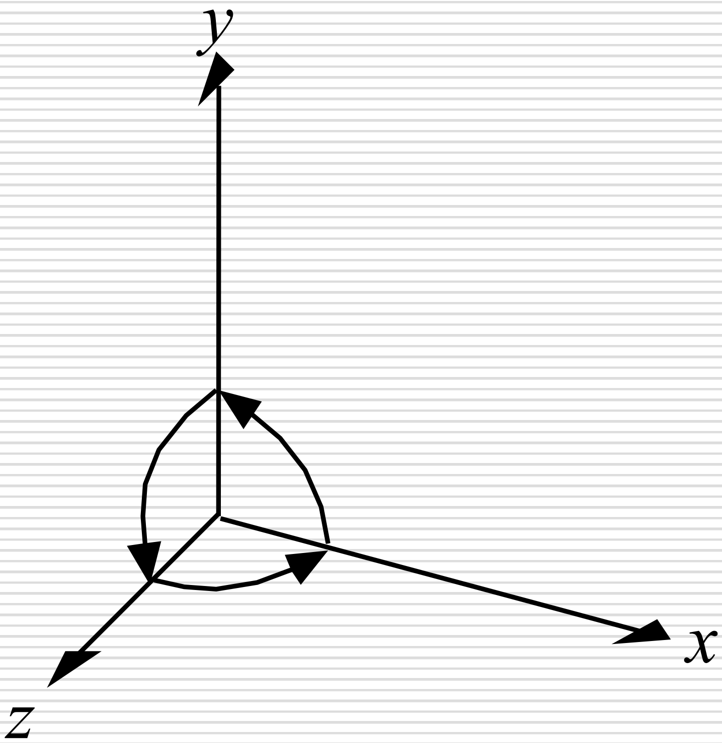
$P_1 = (x_1, y_1)$ $\quad T(-x_1, -y_1)$ $\quad R(\theta)$ $\quad T(x_1, y_1)$

Composition of 2D Transformations

$$T(x_1, y_1) \bullet R(\theta) \bullet T(-x_1, -y_1) = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_1(1-\cos\theta) + y_1 \sin\theta \\ \sin\theta & \cos\theta & y_1(1-\cos\theta) - x_1 \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

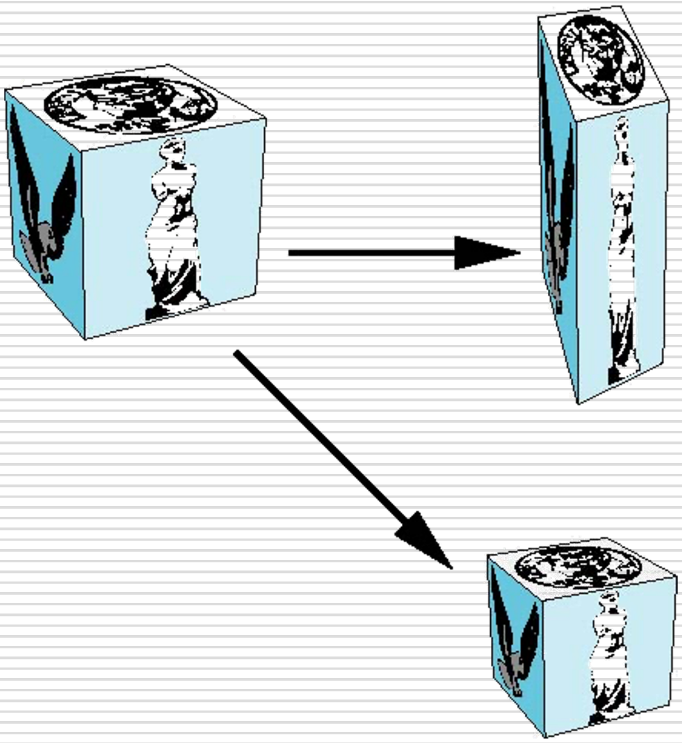
Right-handed Coordinate System



3D Translation & 3D Scaling

$$T(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Reflection & 3D Shearing

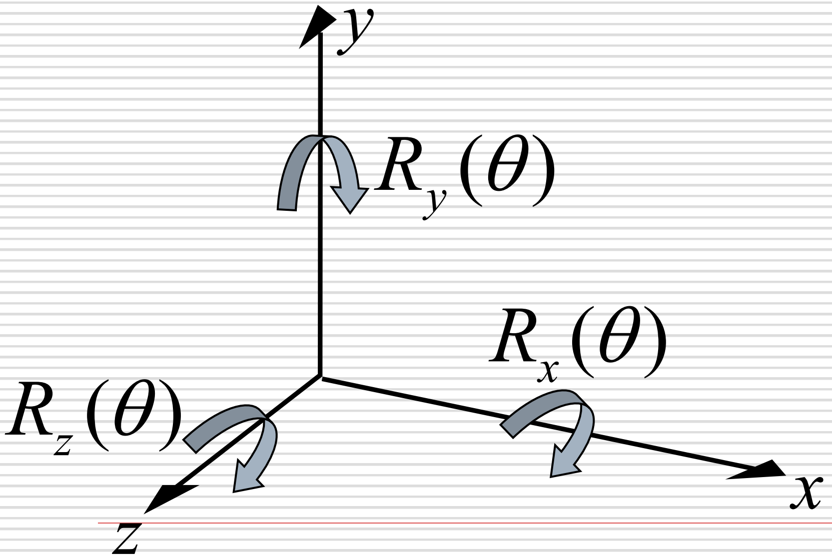
$$RE_x = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad RE_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$SH_{xy}(sh_x, sh_y) = \begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Rotations

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

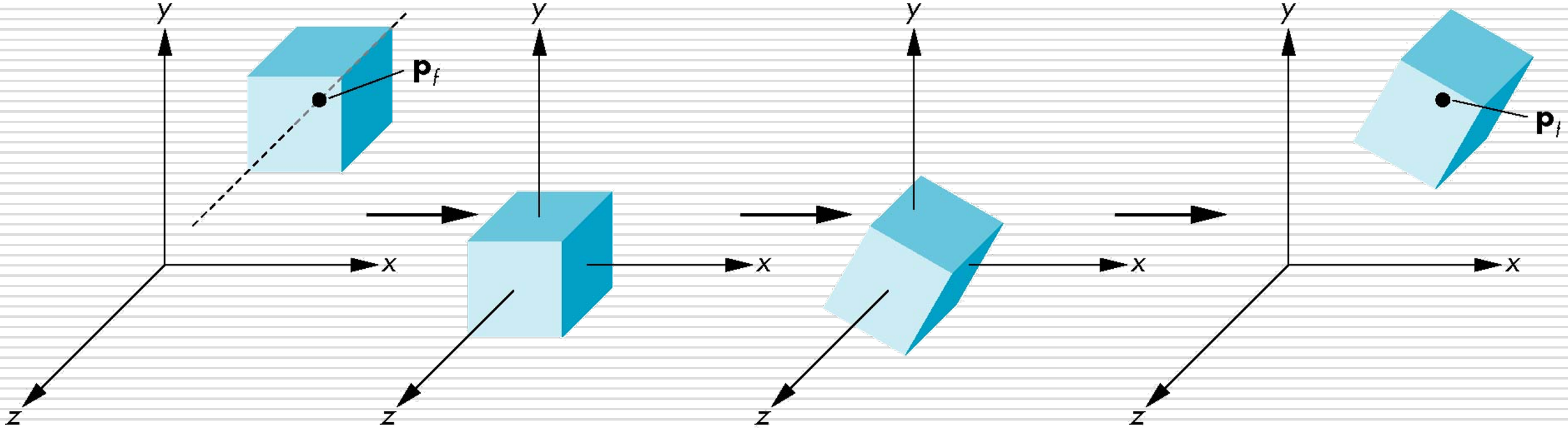
$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

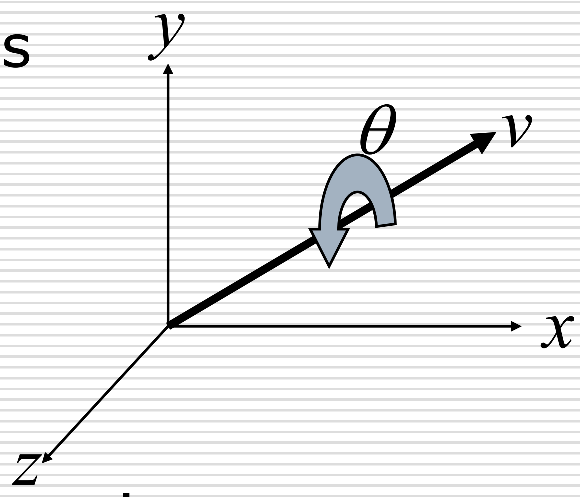
Rotation About a Fixed Point other than the Origin

- Move fixed point to origin
- Rotate
- Move fixed point back
- $M = T(P_f) \bullet R(\theta) \bullet T(-P_f)$



General Rotation About the Origin

- A rotation by θ about an arbitrary axis can be decomposed into the concatenation of rotations about the x, y, and z axes
  - $\theta_x, \theta_y, \theta_z$ are called the Euler angles



$$R(\theta) = R_z(\theta_z) \bullet R_y(\theta_y) \bullet R_x(\theta_x)$$

- Note that rotations do not commute
  - We can use rotations in another order but with different angles.

Transform in Unity

- Transform
  - Every object in a Scene has a Transform.
  - Position, rotation and scale of an object.
  - Hierarchically structure

```
using UnityEngine;
public class Example : MonoBehaviour {
    // Moves all transform children 10 units upwards!
    void Start() {
        foreach (Transform child in transform) {
            child.position += Vector3.up * 10.0f;
        }
    }
}
```

Transform in Unity

- Properties

forward	The blue axis of the transform in world space.
right	The red axis of the transform in world space.
up	The green axis of the transform in world space.
position	The position of the transform in world space.
rotation	The rotation of the transform in world space stored as a Quaternion.
lossyScale	The global scale of the object (Read Only).
localPosition	Position of the transform relative to the parent transform.
localRotation	The rotation of the transform relative to the transform rotation of the parent.
localScale	The scale of the transform relative to the parent.

Transform in Unity

- Properties

root	Returns the topmost transform in the hierarchy.
parent	The parent of the transform.
childCount	The number of children the parent Transform has.
localToWorldMatrix	Matrix that transforms a point from local space into world space (Read Only).
worldToLocalMatrix	Matrix that transforms a point from world space into local space (Read Only).

Transform in Unity

- Public Methods

Translate	Moves the transform in the direction and distance of translation.
Rotate	Applies a rotation of eulerAngles.z degrees around the z axis, eulerAngles.x degrees around the x axis, and eulerAngles.y degrees around the y axis (in that order).
RotateAround	Rotates the transform about axis passing through point in world coordinates by angle degrees.
LookAt	Rotates the transform so the forward vector points at /target/'s current position.

Transform in Unity

- Public Methods

InverseTransformVector	Transforms a vector from world space to local space. The opposite of Transform.TransformVector.
SetPositionAndRotation	Sets the world space position and rotation of the Transform component.
TransformDirection	Transforms direction from local space to world space.
TransformPoint	Transforms position from local space to world space.
TransformVector	Transforms vector from local space to world space.

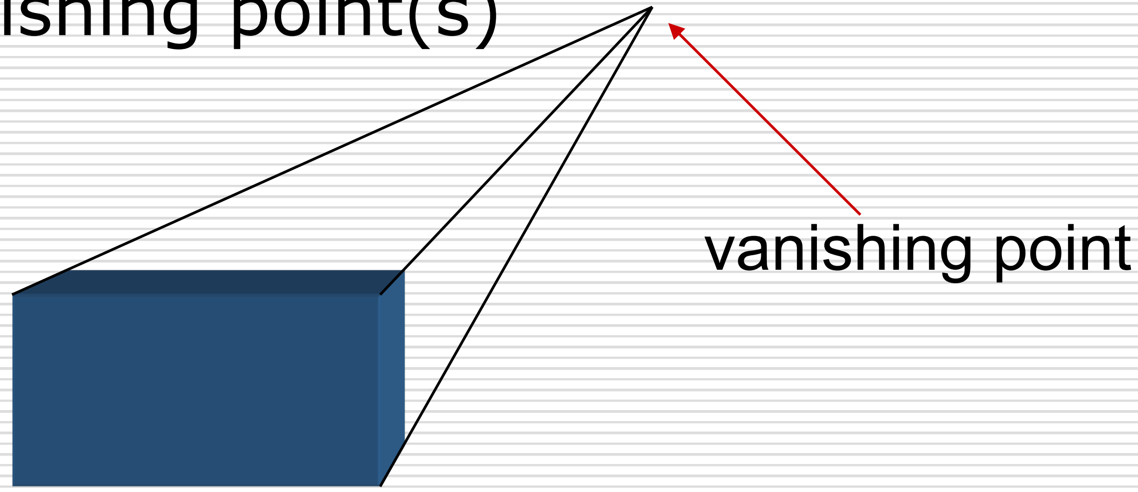
Transform in Unity

- Public Methods

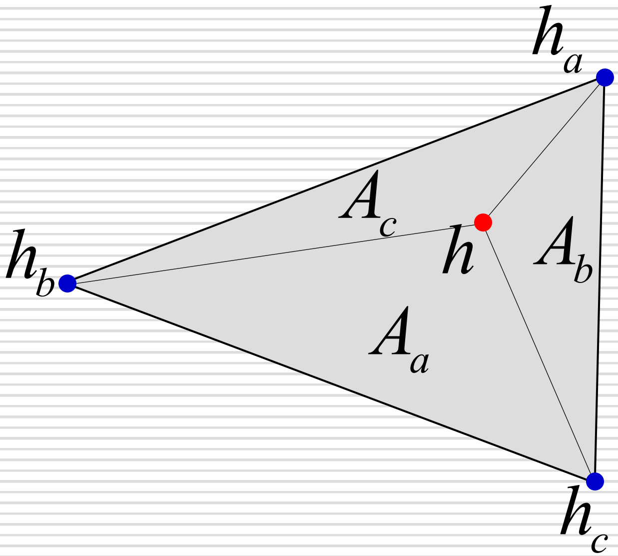
DetachChildren	Unparents all children.
Find	Finds a child by n and returns it.
GetChild	Returns a transform child by index.
GetSiblingIndex	Gets the sibling index.
IsChildOf	Is this transform a child of parent?
SetAsFirstSibling	Move the transform to the start of the local transform list.
SetAsLastSibling	Move the transform to the end of the local transform list.
SetParent	Set the parent of the transform.
SetSiblingIndex	Sets the sibling index.

Vanishing Points

- Parallel lines (not parallel to the projection plan) on the object converge at a single point in the projection (the *vanishing point*)
- Drawing simple perspectives by hand uses these vanishing point(s)



Triangular Coordinate System



$$h = \frac{A_a}{A} h_a + \frac{A_b}{A} h_b + \frac{A_c}{A} h_c$$

where $A = A_a + A_b + A_c$

if $(A_a < 0 \| A_b < 0 \| A_c < 0)$ than
the point is outside the triangle

Triangular Coordinate System - Application

- Terrain following
  - Interpolating the height of arbitrary point within the triangle
- Hit test
  - Intersection of a ray from camera to a screen position with a triangle
- Ray cast
  - Intersection of a ray with a triangle
- Collision detection
  - Intersection

Intersection

- Ray cast
- Containment test

Ray Cast – The Ray

- Cast a ray to calculate the intersection of the ray with models
- Use parametric equation for a ray

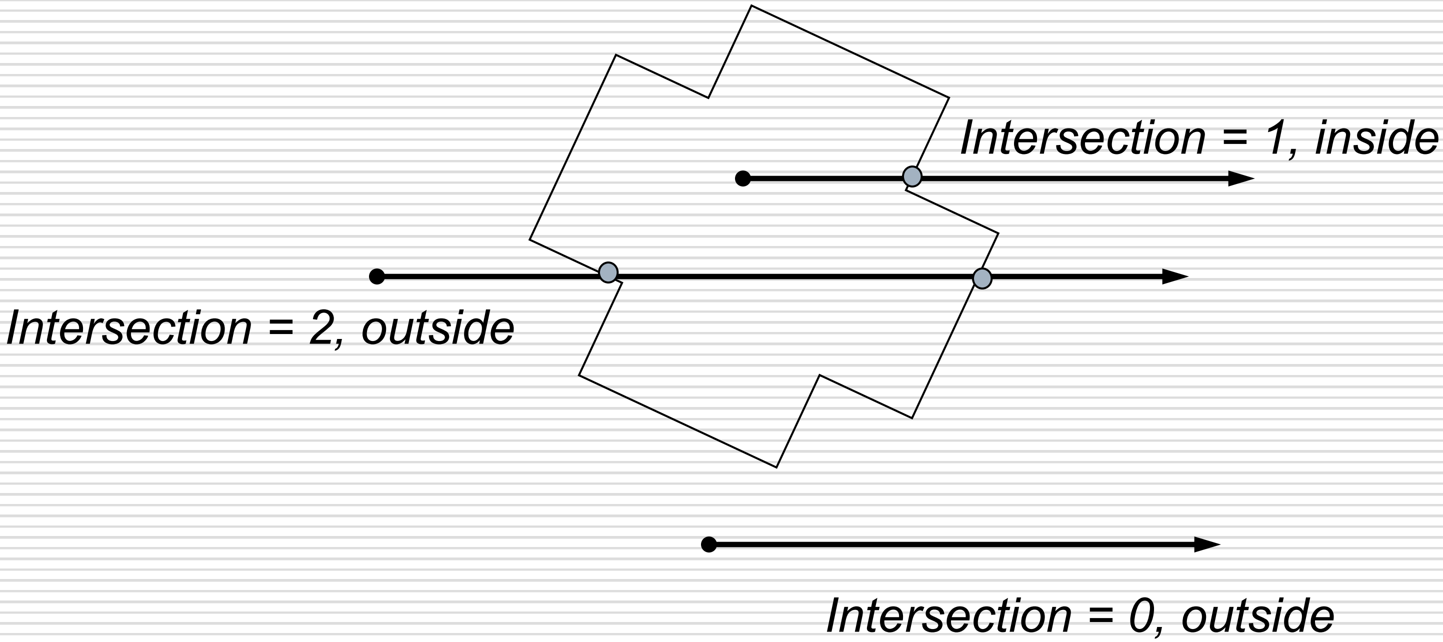
$$\begin{cases} x = x_0 + (x_1 - x_0)t, \\ y = y_0 + (y_1 - y_0)t, \\ z = z_0 + (z_1 - z_0)t, \quad \mathrm{t} \geq 0 \end{cases}$$

- When $t = 0$, the ray is on the start point (x_0, y_0, z_0)
- Only the $t \geq 0$ is the answer candidate
- The smallest positive t is the answer

Ray Cast – The Plane

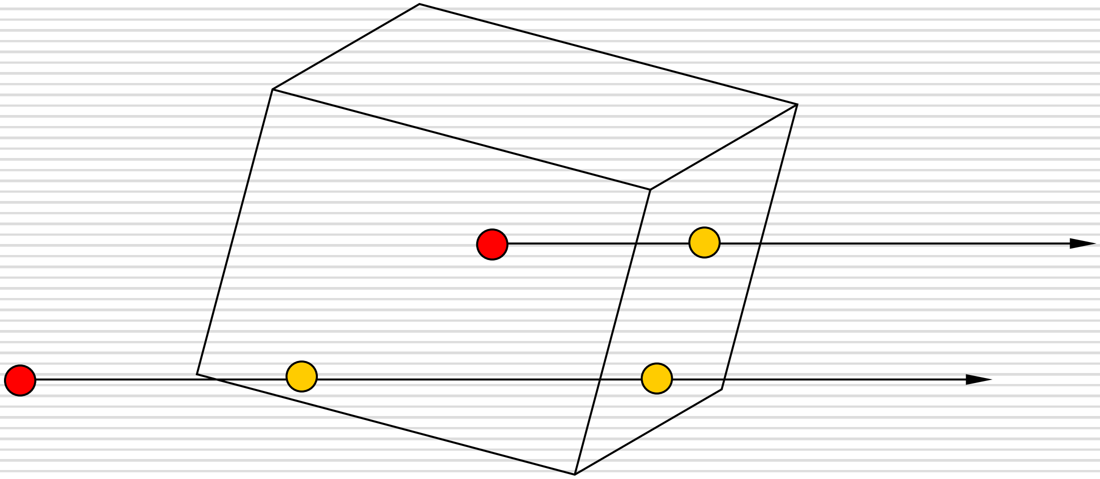
- Each triangle in the 3D models has its plane equation.
- Use $Ax + By + Cz + D = 0$ as the plane equation.
- (A, B, C) is the plane normal vector.
- $|D|$ is the distance of the plane to origin.
- Substitute the ray equation into the plane.
- Solve the t to find the intersect point.
- Check the intersect point within the triangle or not by using "Triangle Area Test".

2D Containment Test



- if the no. of intersection is odd, the point is inside, otherwise, is outside

3D Containment Test



- if the no. of intersection is odd, the point is inside, otherwise, is outside

Ray Cast in Unity

- to the closest object

目前物件的位置作為起點

方向

儲存擊中的結果

```
RaycastHit hitInfo = new RaycastHit();
Vector3 dir = new Vector3(-1,0,0);
if(Physics.Raycast(this.transform.position, dir, out hitInfo)) {
    if (hitInfo.collider.gameObject.name == "CubeA") {
        print("shoot");
    }
}
```

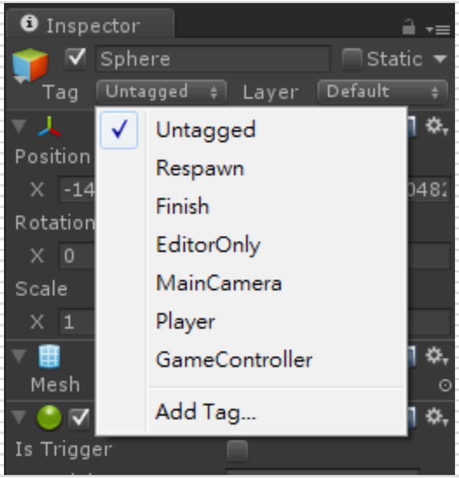
Ray Cast in Unity

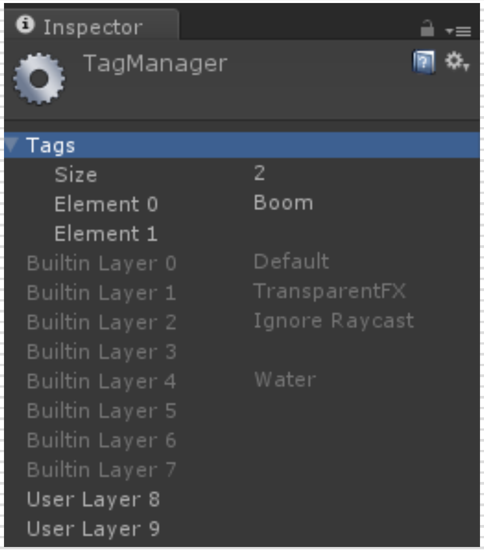
- to all objects

```
Vector3 dir = new Vector3(-1,0,0);
RaycastHit[] hitInfos =
Physics.RaycastAll(this.transform.position, dir);
foreach (RaycastHit hitInfo in hitInfos) {
    print(hitInfo.collider.gameObject.name);
}
```

Tagging Objects for Ray Cast in Unity

- Tag旗標是用來識別和分類物件的方法之一
- 在Ray Cast時能用於識別物件
- 設定Tag的方法在Inspector視窗中的Tag中點擊Add Tag...之後會出現TagManager, 拉下Tags之後就能加入新的Tag





Ray Cast for a Tag in Unity

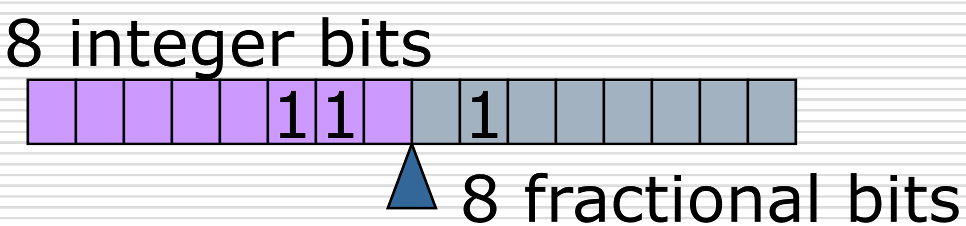
```
Vector3 dir = new Vector3(-1,0,0);
RaycastHit[] hitInfos =
Physics.RaycastAll(this.transform.position, dir);
foreach (RaycastHit hitInfo in hitInfos) {
   if (hitInfo.collider.gameObject.tag != "Boom")
      print(hitInfo.collider.gameObject.name);
}
```

Ray Cast from Camera's view in Unity

```
RaycastHit hit = new RaycastHit();
Vector3 pos = Input.mousePosition;
Ray mouseray = Camera.main.ScreenPointToRay(pos);
if (Input.GetMouseButton(0)) {
    if (Physics.Raycast(mouseray,out hit) ) {
        ……
    }
}
```

Fixed Point Arithmetic

- Fixed point arithmetic: *n* bits (signed) integer
  - Example : *n* = 16 gives range
    $$-32768 \le \tilde{a} \le 32767$$
  - We can use fixed scale to get the decimals.



$$a = \tilde{a} \bullet 2^{-8}$$

$$\tilde{a} = 1600 \rightarrow a = 6.25$$

Fixed Point Arithmetic

- Multiplication requires rescaling

$$e = a \bullet c = \tilde{a} \bullet 2^{-8} \bullet \tilde{c} \bullet 2^{-8} = \tilde{e} \bullet 2^{-8}$$

$$\Rightarrow \tilde{e} = (\tilde{a} \bullet \tilde{c}) \bullet 2^{-8}$$

- Addition just like normal

$$e = a + c = \tilde{a} \bullet 2^{-8} + \tilde{c} \bullet 2^{-8} = (\tilde{a} + \tilde{c}) \bullet 2^{-8}$$

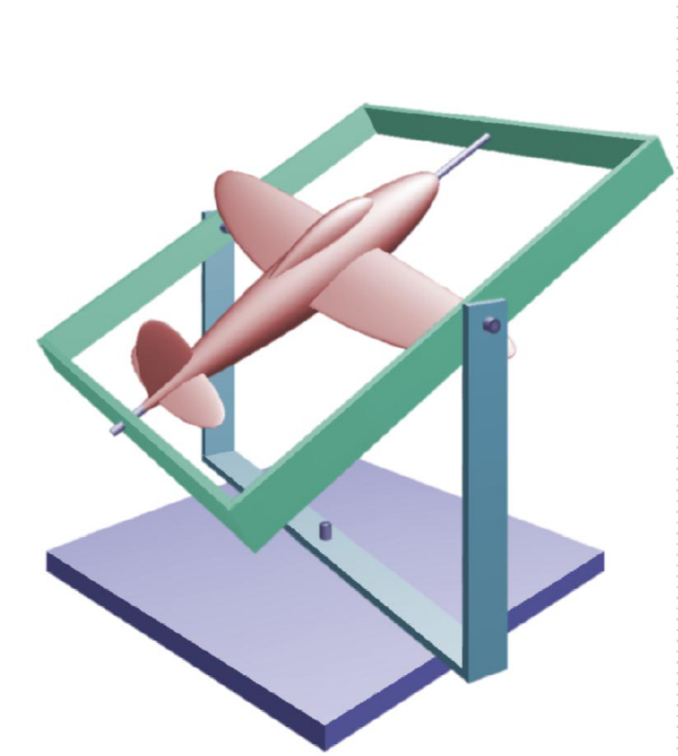
$$\Rightarrow \tilde{e} = \tilde{a} + \tilde{c}$$

Fixed Point Arithmetic - Application

- Compression for floating-point real numbers
  - 4 bytes reduced to 2 bytes
  - Lost some accuracy but affordable
- Network data transfer
- Software 3D rendering (without hardware-assistant)

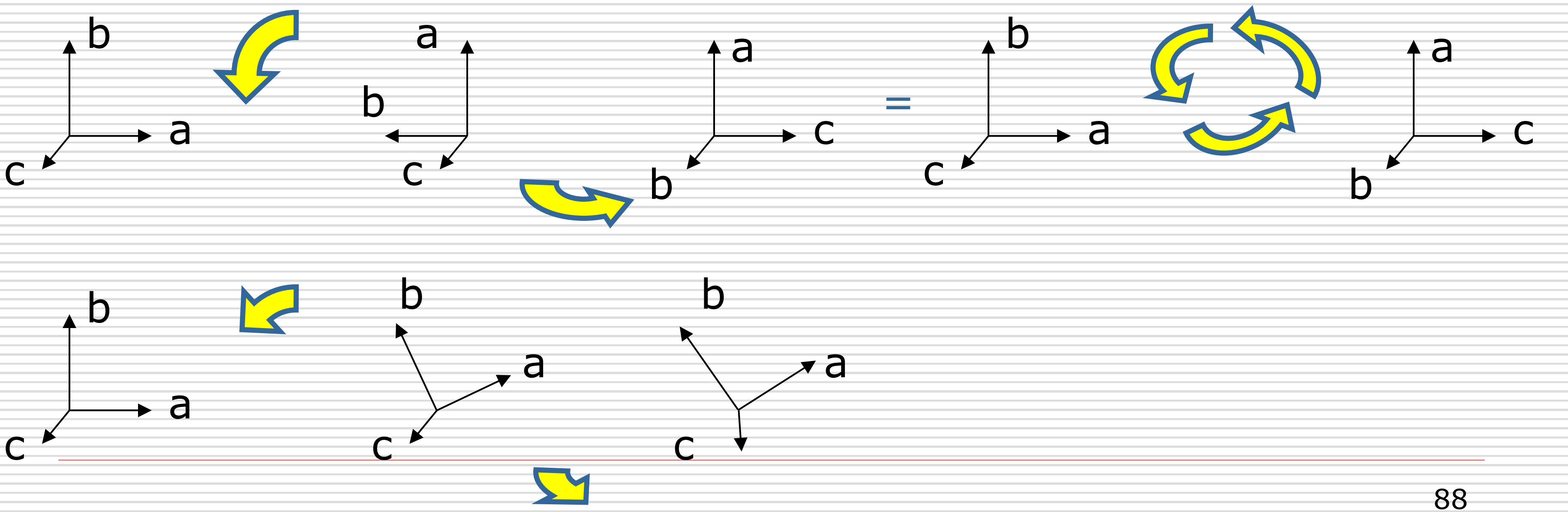
Euler Angles

- An Euler angle is a rotation about a single axis.
- A rotation is described as a sequence of rotations about three mutually orthogonal coordinates axes fixed in space
  - X-roll, Y-roll, Z-roll
- There are 6 possible ways to define a rotation.
  - 3!



Interpolating Euler Angles

- Natural orientation representation:
  - 3 angles for 3 degrees of freedom
- Unnatural interpolation:
  - A rotation of 90^o first around Z and then around Y = 120^o around (1, 1, 1).
  - But 30^o around Z then Y differs from 40^o around (1, 1, 1).

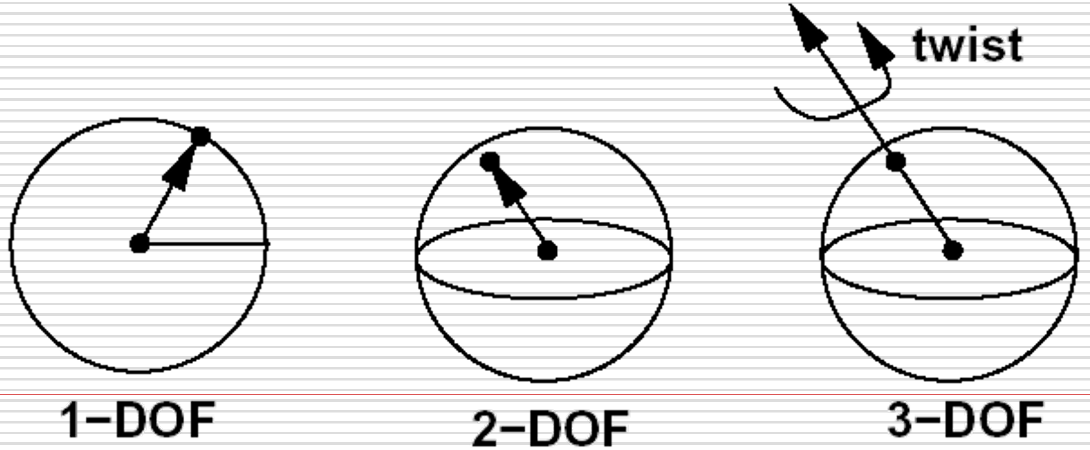


Incremental Rotation

- Consider the two approaches
  - For a sequence of rotation matrices $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_n$, find the Euler angles for each and use $\mathbf{R}_i = \mathbf{R}_{iz}\mathbf{R}_{iy}\mathbf{R}_{ix}$
    - Not very efficient
  - Use the final positions to determine the axis and angle of rotation, then increment only the angle
- Quaternions can be more efficient than either

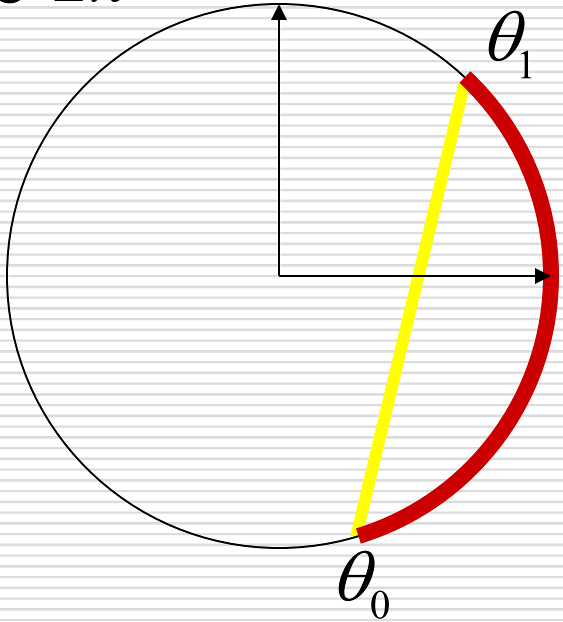
Solution: Quaternion Interpolation

- Interpolate orientation on the unit sphere
- By analogy: 1-, 2-, 3-DOF rotations as constrained points on 1-, 2-, 3-spheres

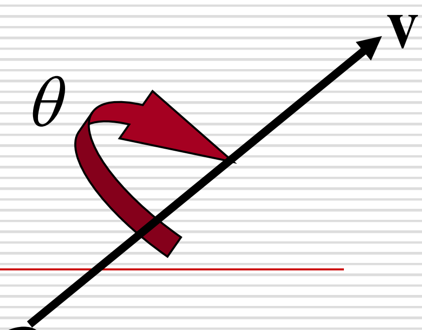


1D-Sphere and Complex Plane

- Interpolate orientation in 2D
- 1 angle
  - but messy because modulo 2π
- Use interpolation in (complex) 2D plane
- Orientation = complex argument of the number



Quaternions



- Quaternions are unit vectors on 3-sphere (in 4D)
- Right-hand rotation of θ radians about $\mathbf{v}$ is $q = [\cos(\theta/2), \sin(\theta/2) \bullet \mathbf{v}]$
  - often noted $[\mathbf{w}, \mathbf{v}]$
- Requires one real and three imaginary components $\mathbf{i}$, $\mathbf{j}$,$\mathbf{k}$
  - $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = [\mathbf{w}, \mathbf{v}]; \mathbf{w} = q_0, \mathbf{v} = (q_1, q_2, q_3)$
  - where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
  - $\mathbf{w}$ is called **scalar** and $\mathbf{v}$ is called **vector**

Basic Operations Using Quaternions

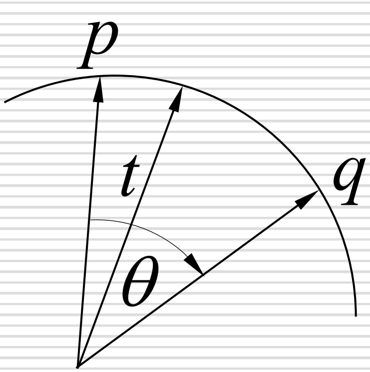
- Addition $q + q' = [\mathbf{w} + \mathbf{w}', \mathbf{v} + \mathbf{v}']$
- Multiplication $q \bullet q' = [\mathbf{w} \bullet \mathbf{w}' - \mathbf{v} \bullet \mathbf{v}', \mathbf{v} \times \mathbf{v}' + \mathbf{w} \bullet \mathbf{v}' + \mathbf{w}' \bullet \mathbf{v}]$
- Conjugate $q^* = [\mathbf{w}, -\mathbf{v}]$
- Length $|q| = (\mathbf{w}^2 + |\mathbf{v}|^2)^{1/2}$
- Norm $\mathrm{N}(q) = |q|^2 = \mathbf{w}^2 + |\mathbf{v}|^2$
- Inverse $q^{-1} = q^* / |q|^2 = q^* / \mathrm{N}(q)$
- Unit Quaternion
  - q is a unit quaternion if $|q| = 1$ and then $q^{-1} = q^*$
- Identity
  - [1, (0, 0, 0)] (when involving multiplication)
  - [0, (0, 0, 0)] (when involving addition)

SLERP-Spherical Linear intERPolation

- Interpolate between two quaternion rotations along the shortest arc.

- $$\mathrm{SLERP}(p,q,t) = \frac{p \bullet \sin((1-t)\bullet\theta) + q \bullet \sin(t \bullet \theta)}{\sin(\theta)}$$

  - where $\cos(\theta) = \mathbf{w}_p \bullet \mathbf{w}_q + \mathbf{v}_p \bullet \mathbf{v}_q$



- If two orientations are too close, use linear interpolation to avoid any divisions by zero.

Quaternion in Unity

- They are compact, don't suffer from gimbal lock and can **easily be interpolated**. Unity internally uses Quaternions to represent all rotations.
- never access or modify individual Quaternion components (x,y,z,w)
- 99% of the time are:
  - Quaternion.LookRotation
  - Quaternion.Angle
  - Quaternion.Euler
  - Quaternion.Slerp
  - Quaternion.FromToRotation
  - Quaternion.identity

Parametric Polynomial Curves

- We will use parametric curves where the functions are all polynomials in the parameter.

$$x(u) = \sum_{k=0}^{n} a_k u^k$$

$$y(u) = \sum_{k=0}^{n} b_k u^k$$

- Advantages:
  - easy (and efficient) to compute
  - infinitely differentiable

Parametric Cubic Curves

- Fix $n = 3$
- The cubic polynomials that define a curve segment $Q(t) = [x(t) \quad y(t) \quad z(t)]^T$ are of the form

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x,$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y,$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z, \quad 0 \le t \le 1.$$

Parametric Cubic Curves

- The curve segment can be rewrite as

$$Q(t) = [x(t) \quad y(t) \quad z(t)]^{\mathrm{T}} = C \bullet T$$

- where $T = [t^3 \quad t^2 \quad t \quad 1]^{\mathrm{T}}$

$$C = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix}$$

Tangent Vector

$$\frac{d}{dt}Q(t) = Q'(t) = \left[\frac{d}{dt}x(t) \quad \frac{d}{dt}y(t) \quad \frac{d}{dt}z(t)\right]^{\mathrm{T}}$$

$$= \frac{d}{dt}C \bullet T = C \bullet \left[3t^2 \quad 2t \quad 1 \quad 0\right]^{\mathrm{T}}$$

$$= \left[3a_x t^2 + 2b_x t + c_x \quad 3a_y t^2 + 2b_y t + c_y \quad 3a_z t^2 + 2b_z t + c_z\right]^{\mathrm{T}}$$

Three Types of Parametric Cubic Curves

- Hermite Curves
  - defined by two **endpoints** and two endpoint **tangent vectors**
- Bézier Curves
  - defined by two **endpoints** and two **control points** which control the endpoint' **tangent vectors**
- Splines
  - defined by four **control points**

Parametric Cubic Curves

- $Q(t) = C \bullet T$
- rewrite the coefficient matrix as $C = G \bullet M$
  - where M is a 4x4 **basis matrix**, G is called the **geometry matrix**
  - so

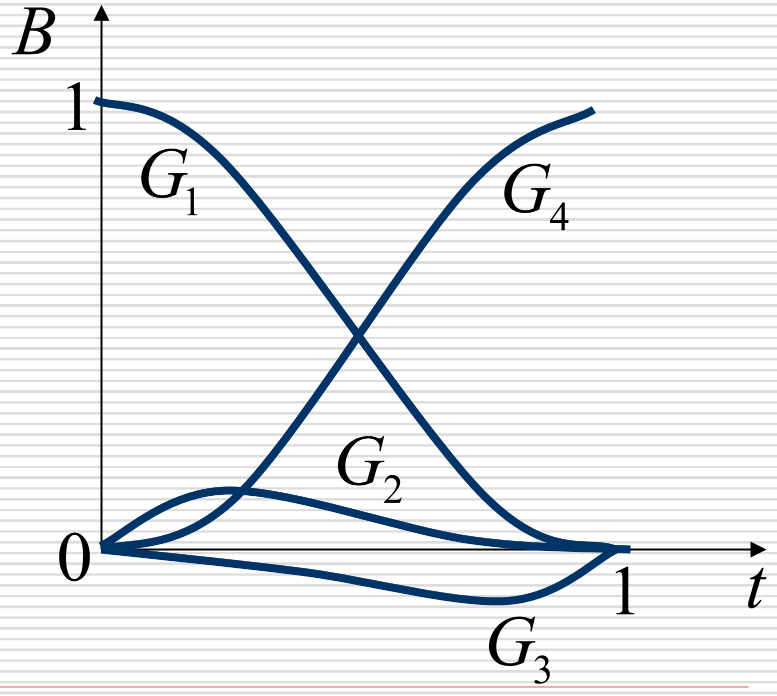
$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} G_1 & G_2 & G_3 & G_4 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

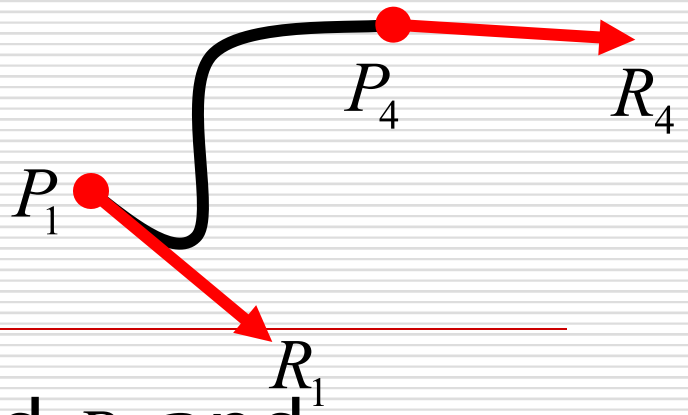
4 endpoints or tangent vectors

Parametric Cubic Curves

- $Q(t) = G \bullet M \bullet T = G \bullet B$

  where $B = M \bullet T$ is called the **blending functions**



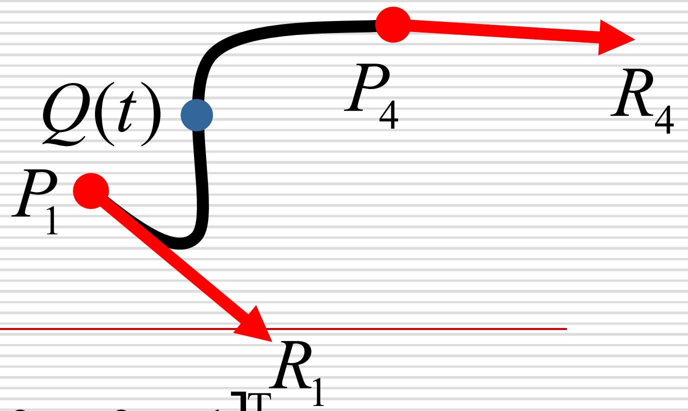


Hermite Curves

- Given the endpoints P_1 and P_4 and tangent vectors at them R_1 and R_4
- What is
  - **Hermite basis matrix** M_{H}
  - **Hermite geometry vector** G_{H}
  - **Hermite blending functions** B_{H}
- by definition

$$G_{\mathrm{H}} = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix}$$

Hermite Curves



- since $Q(0) = P_1 = G_H \bullet M_H \bullet \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$

$$Q(1) = P_4 = G_H \bullet M_H \bullet \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$$Q'(0) = R_1 = G_H \bullet M_H \bullet \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$Q'(1) = R_4 = G_H \bullet M_H \bullet \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix}^T$$

$$G_H = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix} = G_H \bullet M_H \bullet \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Hermite Curves

- so

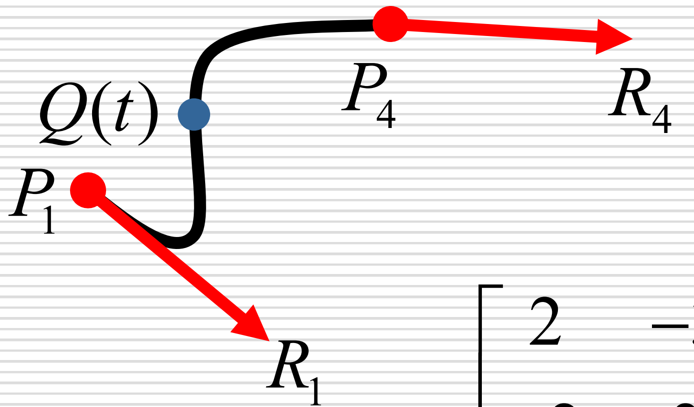
$$M_{\mathrm{H}} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

- and $Q(t) = G_{\mathrm{H}} \bullet M_{\mathrm{H}} \bullet T = G_{\mathrm{H}} \bullet B_{\mathrm{H}}$

$$B_{\mathrm{H}} = \begin{bmatrix} 2t^3 - 3t^2 + 1 & -2t^3 + 3t^2 & t^3 - 2t^2 + t & t^3 - t^2 \end{bmatrix}^{\mathrm{T}}$$

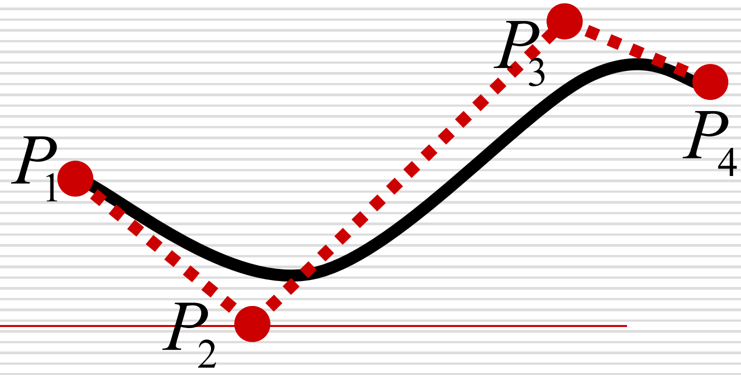
Computing a point

- Given two endpoints P_1 and P_4 and two tangent vectors at them R_1 and R_4



so

$$Q(t) = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$



Bézier Curves

- Given the endpoints P_1 and P_4 and two control points P_2 and P_3 which determine the endpoints' tangent vectors, such that $R_1 = Q'(0) = 3(P_2 - P_1)$

$$R_4 = Q'(1) = 3(P_4 - P_3)$$

- What is
  - **Bézier basis matrix** M_B
  - **Bézier geometry vector** G_B
  - **Bézier blending functions** B_B

Bézier Curves

- by definition $G_B = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}$
- then $G_H = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix}$

$$= \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} = G_B \bullet M_{HB}$$

- so $Q(t) = G_H \bullet M_H \bullet T = (G_B \bullet M_{HB}) \bullet M_H \bullet T$

$$= G_B \bullet (M_{HB} \bullet M_H) \bullet T = G_B \bullet M_B \bullet T$$

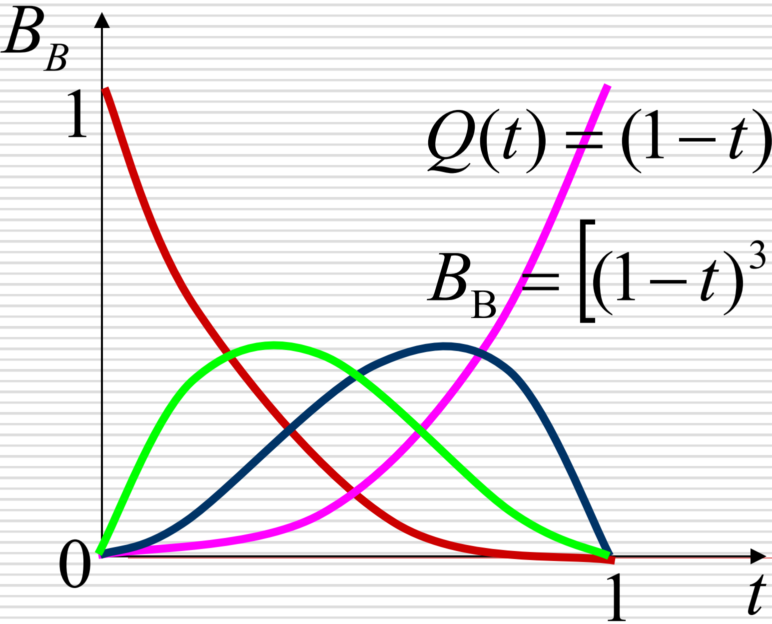
Bézier Curves

- and

$$M_{\mathrm{B}} = M_{\mathrm{HB}} \bullet M_{\mathrm{H}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

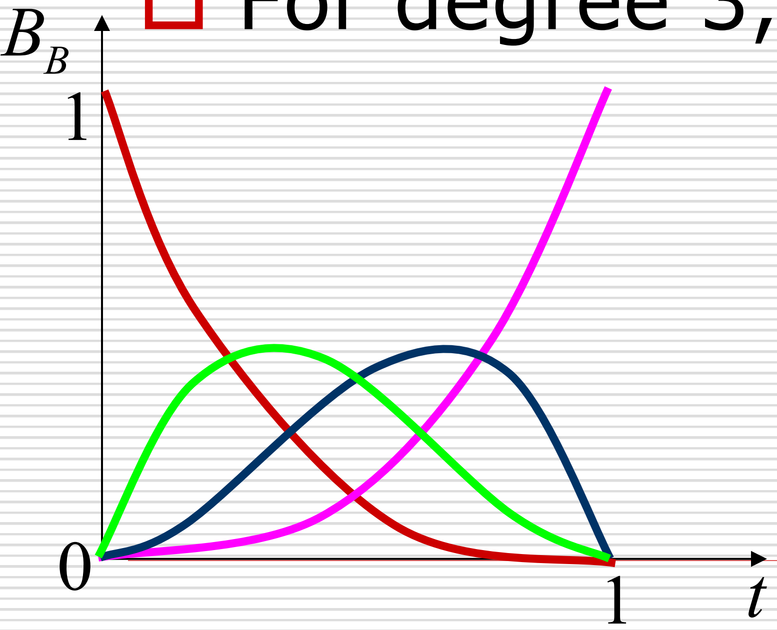
$$B_{\mathrm{B}} = \left[(1-t)^3 \quad 3t(1-t)^2 \quad 3t^2(1-t) \quad t^3 \right]^{\mathrm{T}}$$



Bernstein polynomials

Bernstein Polynomials

- The coefficients of the control points are a set of functions called the **Bernstein polynomials**: $Q(t)=\sum_{i=0}^{n} b_i(t)P_i$
- For degree 3, we have:



$$b_0(t)=(1-t)^3$$

$$b_1(t)=3t(1-t)^2$$

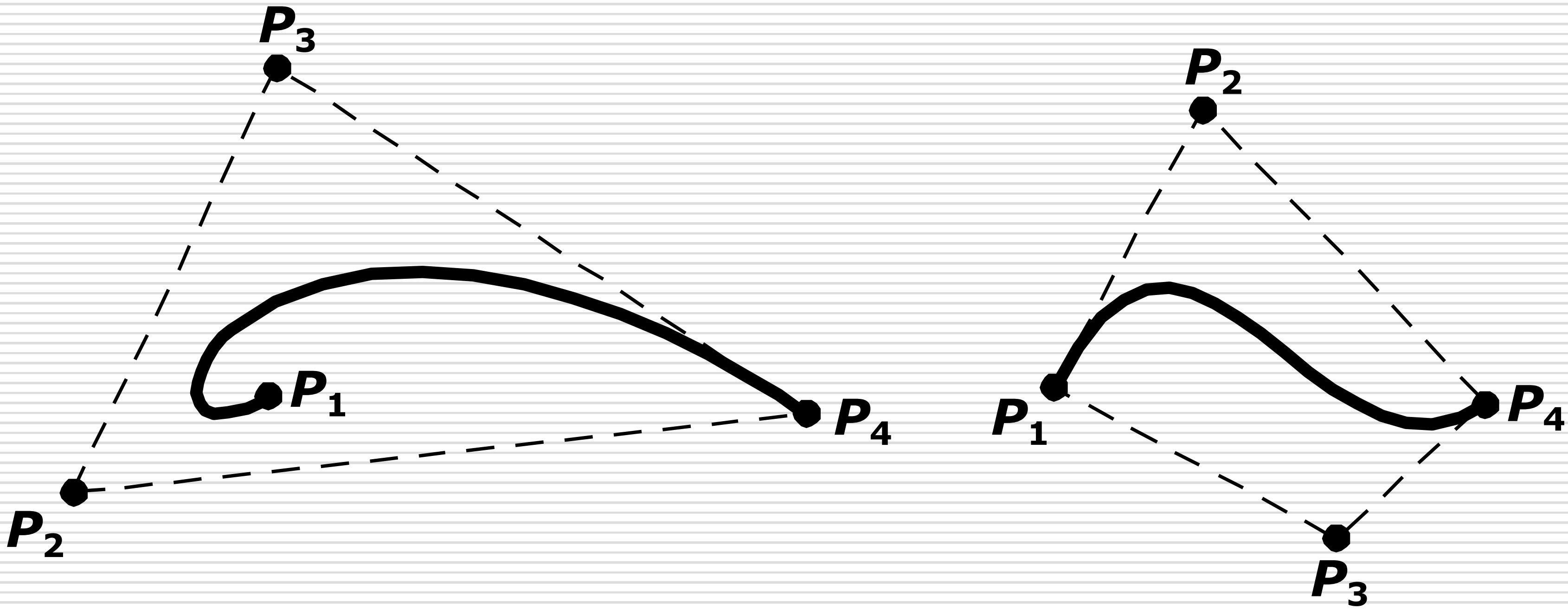
$$b_2(t)=3t^2(1-t)$$

$$b_3(t)=t^3$$

Bernstein Polynomials

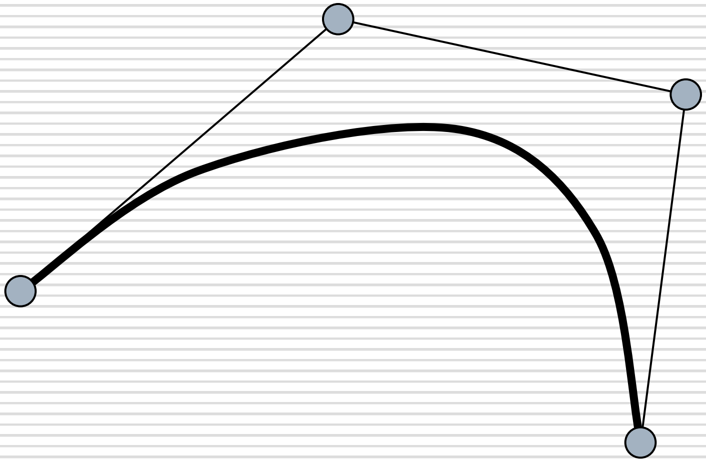
- Useful properties on the interval [0,1]:
  - each is between 0 and 1
  - sum of all four is exact 1
    - a.k.a., a “partition of unity”
- These together imply that the curve lines within the **convex hull** of its control points.

Convex Hull



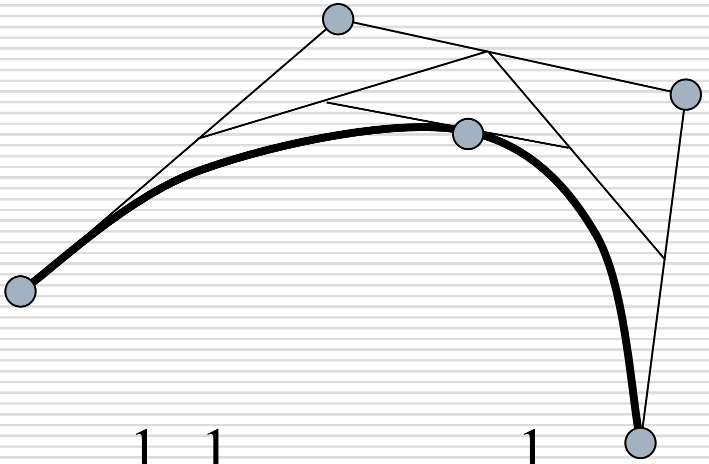
Subdividing Bézier Curves

- $Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$
- How to draw the curve ?
- How to convert it to be line-segments ?



Subdividing Bézier Curves (de Casteljau's algorithm)

- $Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$
- How to draw the curve ?
- How to convert it to be line-segments ?

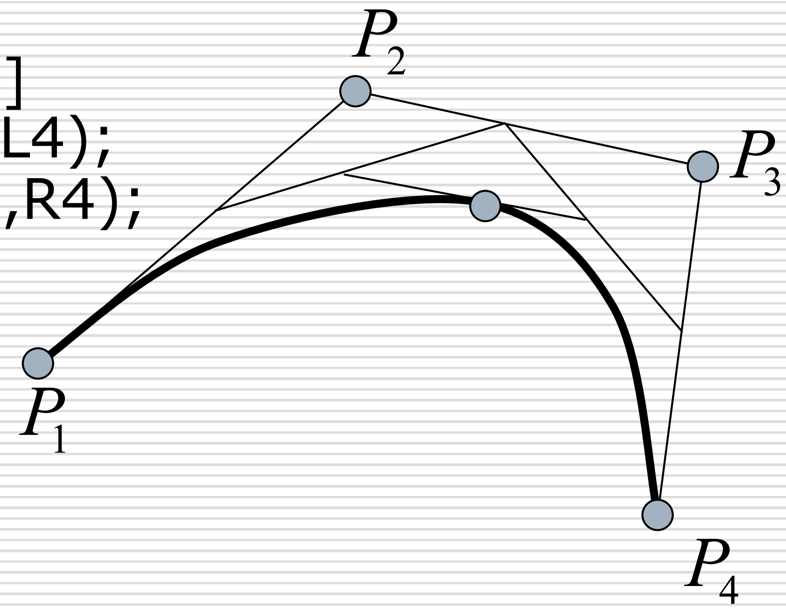


$$Q(\frac{1}{2}) = \frac{1}{8}P_1 + \frac{3}{8}P_2 + \frac{3}{8}P_3 + \frac{1}{8}P_4$$

$$= \frac{1}{2}(\frac{1}{2}(\frac{1}{2}(P_1 + P_2) + \frac{1}{2}(P_2 + P_3)) + \frac{1}{2}(\frac{1}{2}(P_3 + P_4) + \frac{1}{2}(P_2 + P_3)))$$

Display Bézier Curves

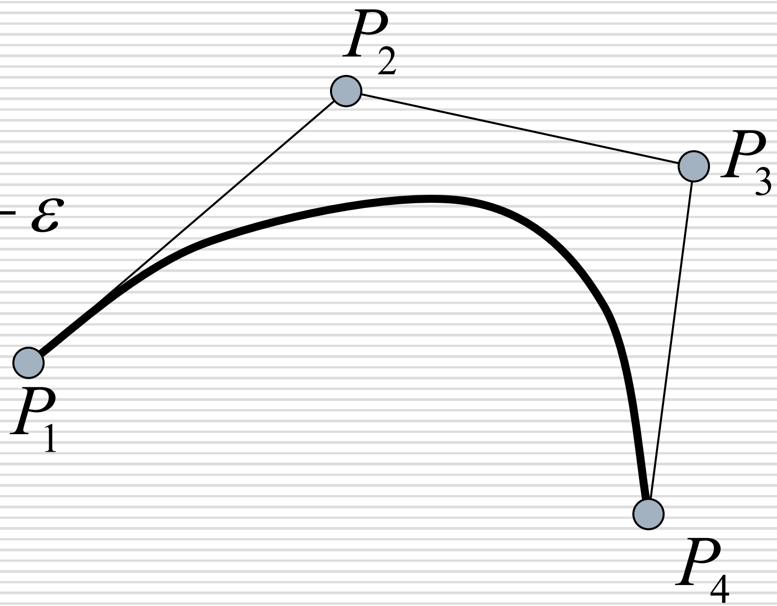
```
DisplayBezier(P1,P2,P3,P4)
begin
    if (FlatEnough(P1,P2,P3,P4))
        Line(P1,P4);
    else
        Subdivide(P[])=>L[],R[]
        DisplayBezier(L1,L2,L3,L4);
        DisplayBezier(R1,R2,R3,R4);
end;
```



Testing for Flatness

- Compare total length of control polygon to length of line connecting endpoints

$$\frac{|P_1 - P_2| + |P_2 - P_3| + |P_3 - P_4|}{|P_1 - P_4|} < 1 + \varepsilon$$

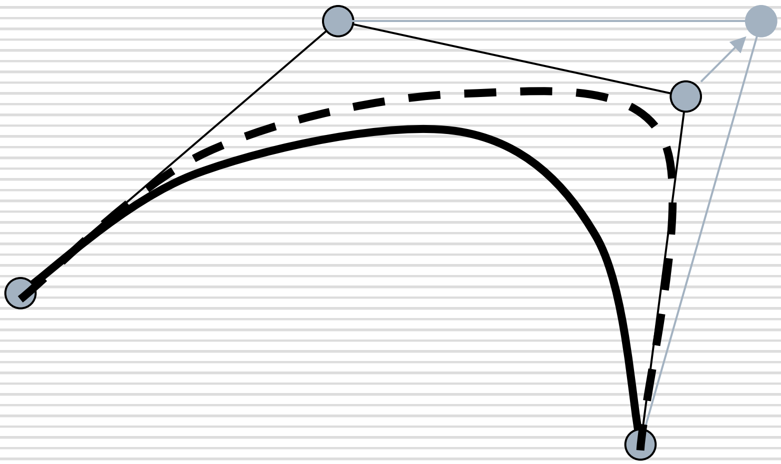


What do we want for a curve?

- Local control
- Interpolation
- Continuity

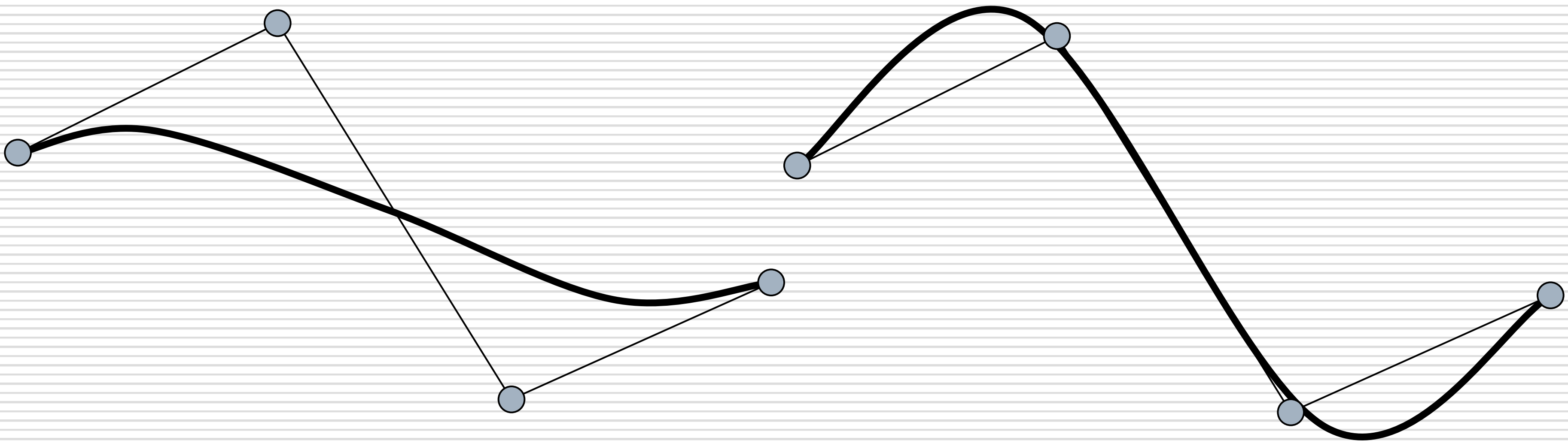
Local Control

- One problem with Bézier curve is that every control points affect every point on the curve (except for endpoints). Moving a single control point affects the whole curve.
- We'd like to have local control, that is, have each control point affect some well-defined neighborhood around that point.

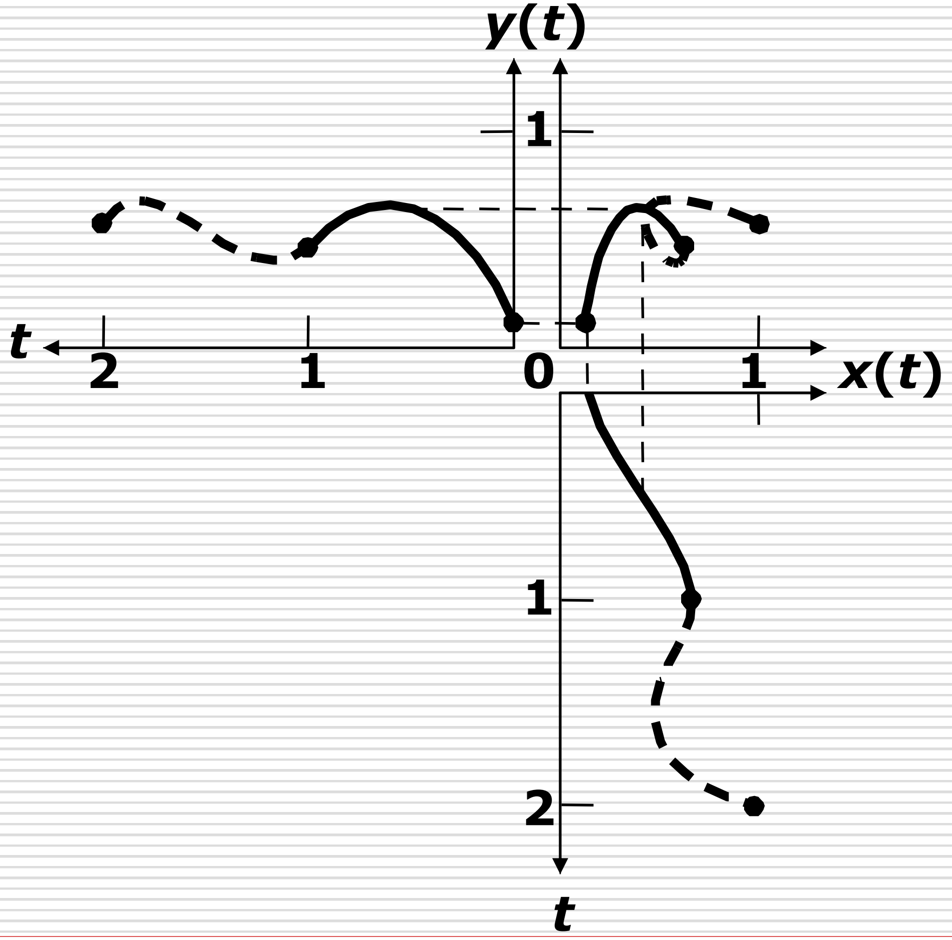


Interpolation

- Bézier curves are approximating. The curve does not necessarily pass through all the control points. We'd like to have a curve that is interpolating, that is, that always passes through every control points.



Continuity between Curve Segments



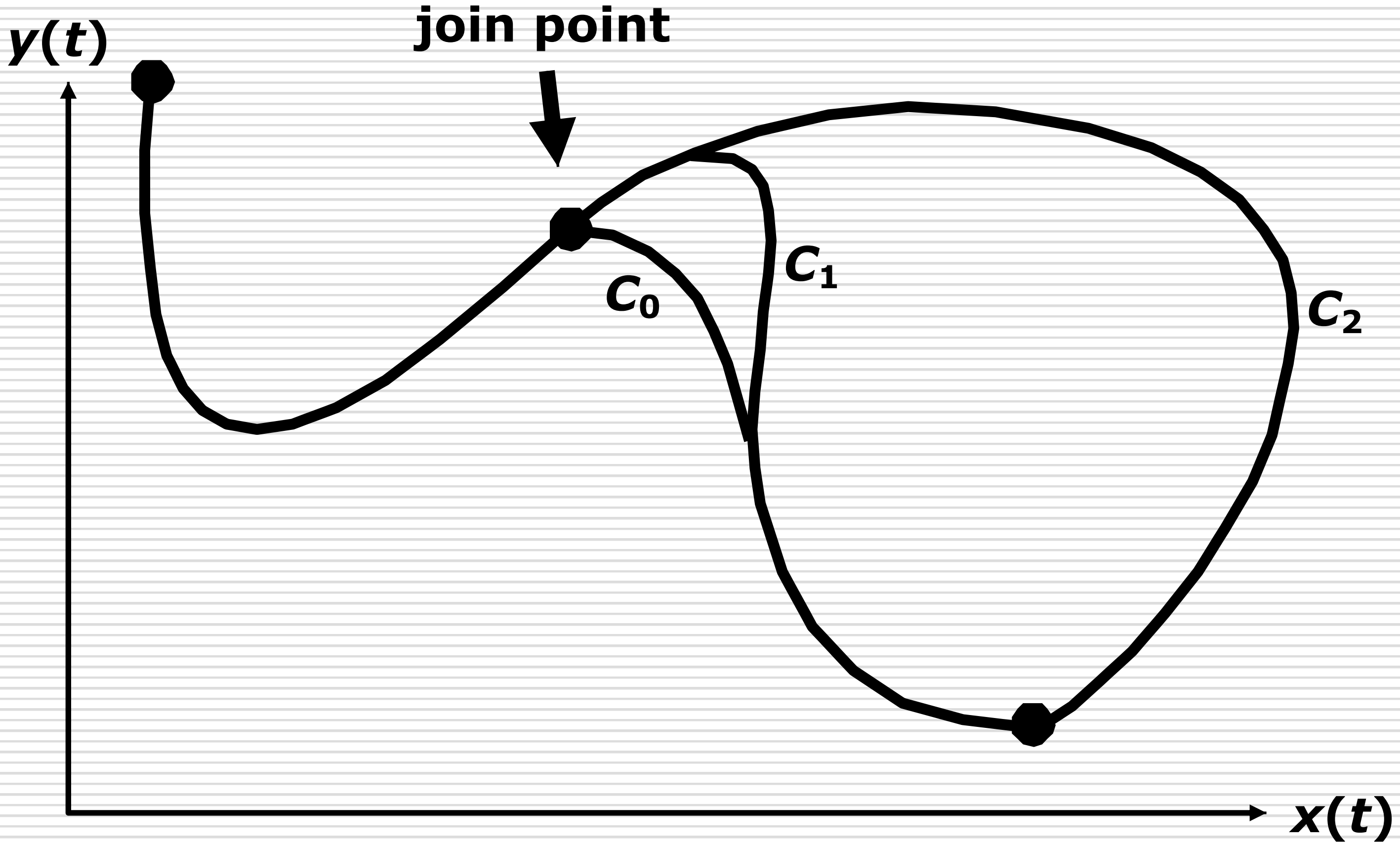
Continuity between Curve Segments

- G^0 geometric continuity
  - two curve segments join together
- G^1 geometric continuity
  - the directions *(but not necessarily the magnitudes)* of the two segments' tangent vectors are equal at a join point

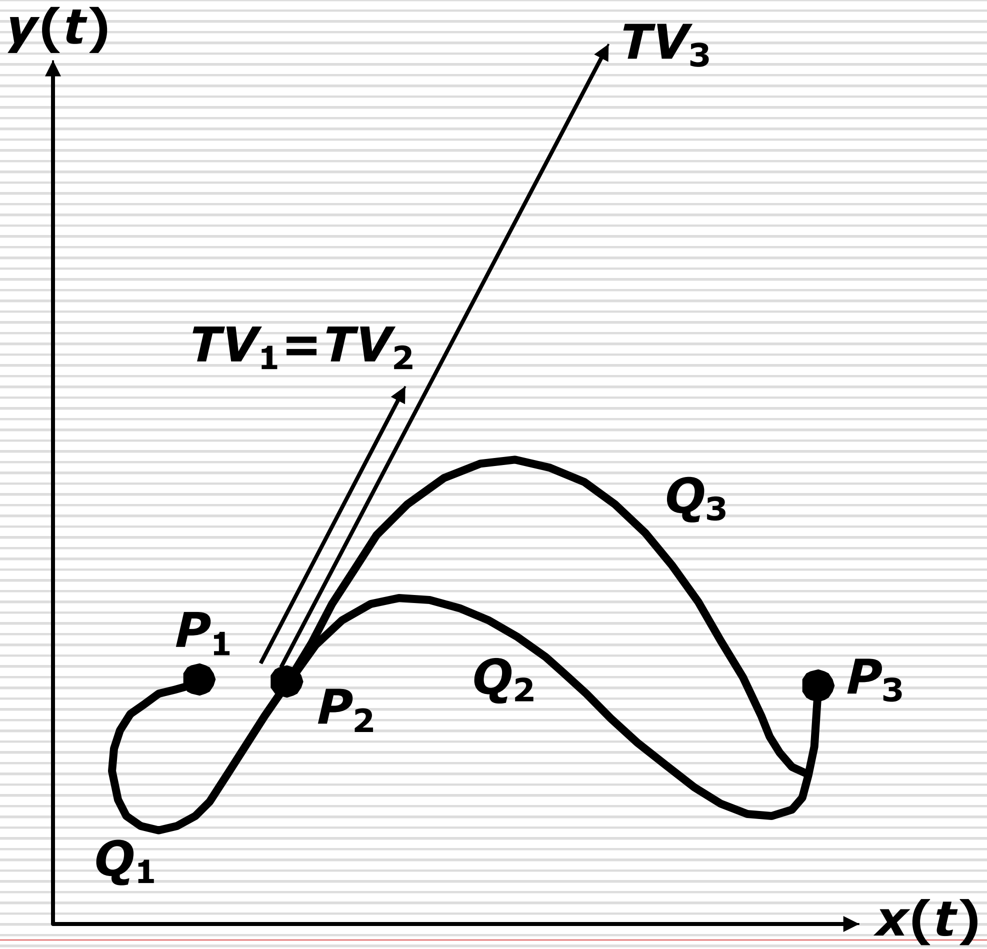
Continuity between Curve Segments

- C^1 continuous
  - the tangent vectors of the two cubic curve segments are equal *(both directions and magnitudes)* at the segments' join point
- C^n continuous
  - the direction and magnitude of $d^n / dt^n [Q(t)]$ through the *n*th derivative are equal at the join point

Continuity between Curve Segments



Continuity between Curve Segments



Bézier Curves → Splines

- Bézier curves have C-infinity continuity on their interiors, but we saw that they do not exhibit local control or interpolate their control points.
- It is possible to define points that we want to interpolate, and then solve for the Bézier control points that will do the job.
- But, you will need as many control points as interpolated points -> high order polynomials -> wiggly curves. (And you still won't have local control.)

Bézier Curves → Splines

- We will splice together a curve from individual Bézier segments. We call these curves **splines**.
- When splicing Bézier together, we need to worry about continuity.

Ensuring C^0 continuity

- Suppose we have a cubic Bézier defined by (V_1, V_2, V_3, V_4), and we want to attach another curve (W_1, W_2, W_3, W_4) to it, so that there is C^0 continuity at the joint.

$$C^0 : Q_V(1) = Q_W(0)$$

- What constraint(s) does this place on (W_1, W_2, W_3, W_4) ?

$$Q_V(1) = Q_W(0) \Rightarrow V_4 = W_1$$

Ensuring C^1 continuity

- Suppose we have a cubic Bézier defined by (V_1, V_2, V_3, V_4), and we want to attach another curve (W_1, W_2, W_3, W_4) to it, so that there is C^1 continuity at the joint.

$$C^0 : Q_V(1) = Q_W(0)$$

$$C^1 : Q'_V(1) = Q'_W(0)$$

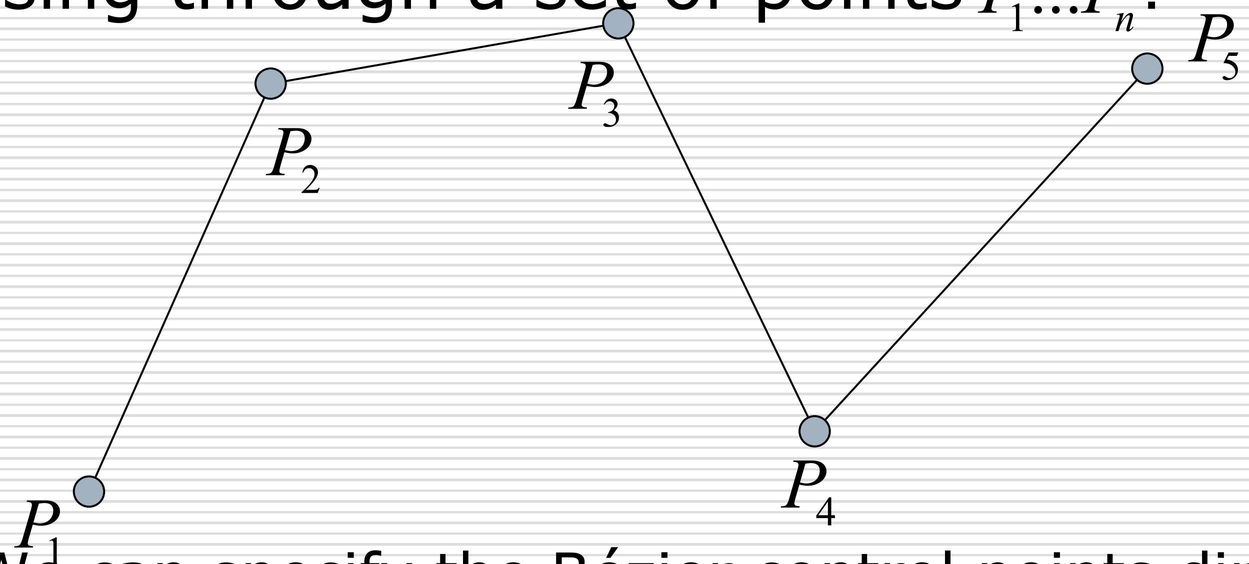
- What constraint(s) does this place on (W_1, W_2, W_3, W_4) ?

$$Q_V(1) = Q_W(0) \Rightarrow V_4 = W_1$$

$$Q'_V(1) = Q'_W(0) \Rightarrow V_4 - V_3 = W_2 - W_1$$

The C^1 Bézier Spline

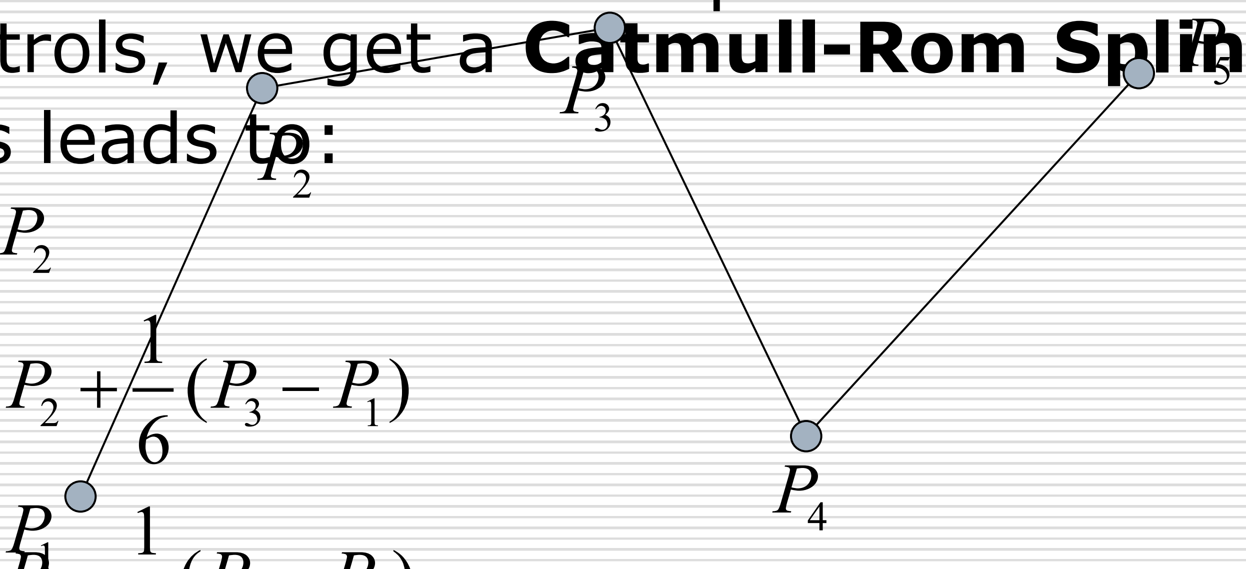
- How then could we construct a curve passing through a set of points $P_1 \ldots P_n$?



- We can specify the Bézier control points directly, or we can devise a scheme for placing them automatically…

Catmull-Rom Spline

- If we set each derivative to be one half of the vector between the previous and next controls, we get a **Catmull-Rom Spline**.
- This leads to:



$$V_1 = P_2$$

$$V_2 = P_2 + \frac{1}{6}(P_3 - P_1)$$

$$V_3 = P_3 - \frac{1}{6}(P_4 - P_2)$$

$$V_4 = P_3$$

Catmull-Rom Basis Matrix

$$Q(t) = G_B \bullet M_B \bullet T$$

$$= G_B \bullet \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet T \quad G_B = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{6} & 1 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 1 & \frac{-1}{6} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

$$Q(t) = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Ensuring C^2 continuity

- Suppose we have a cubic Bézier defined by (V_1, V_2, V_3, V_4), and we want to attach another curve (W_1, W_2, W_3, W_4) to it, so that there is C^2 continuity at the joint.

$$Q_V(1) = Q_W(0) \Rightarrow V_4 = W_1$$

$$Q'_V(1) = Q'_W(0) \Rightarrow V_4 - V_3 = W_2 - W_1$$

$$Q''_V(1) = Q''_W(0) \Rightarrow V_2 - 2V_3 + V_4 = W_1 - 2W_2 + W_3$$

$$\Downarrow$$

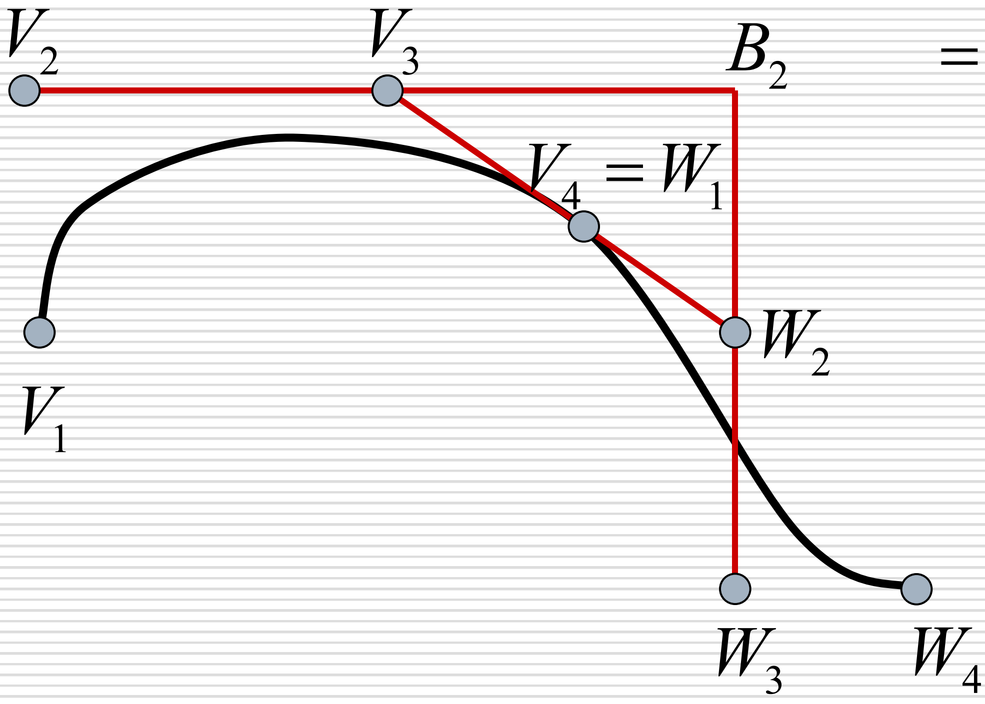
$$W_3 = V_2 - 4V_3 + 4V_4$$

B-Spline

- Instead of specifying the Bézier control points themselves, let's specify the corners of the A-frames in order to build a C^2 continuous spline.

B-Spline

$$W_3 = V_2 - 4V_3 + 4V_4$$
$$= 2(2V_4 - V_3) - (2V_3 - V_2)$$
$$= 2W_2 - B_2$$



B-Spline

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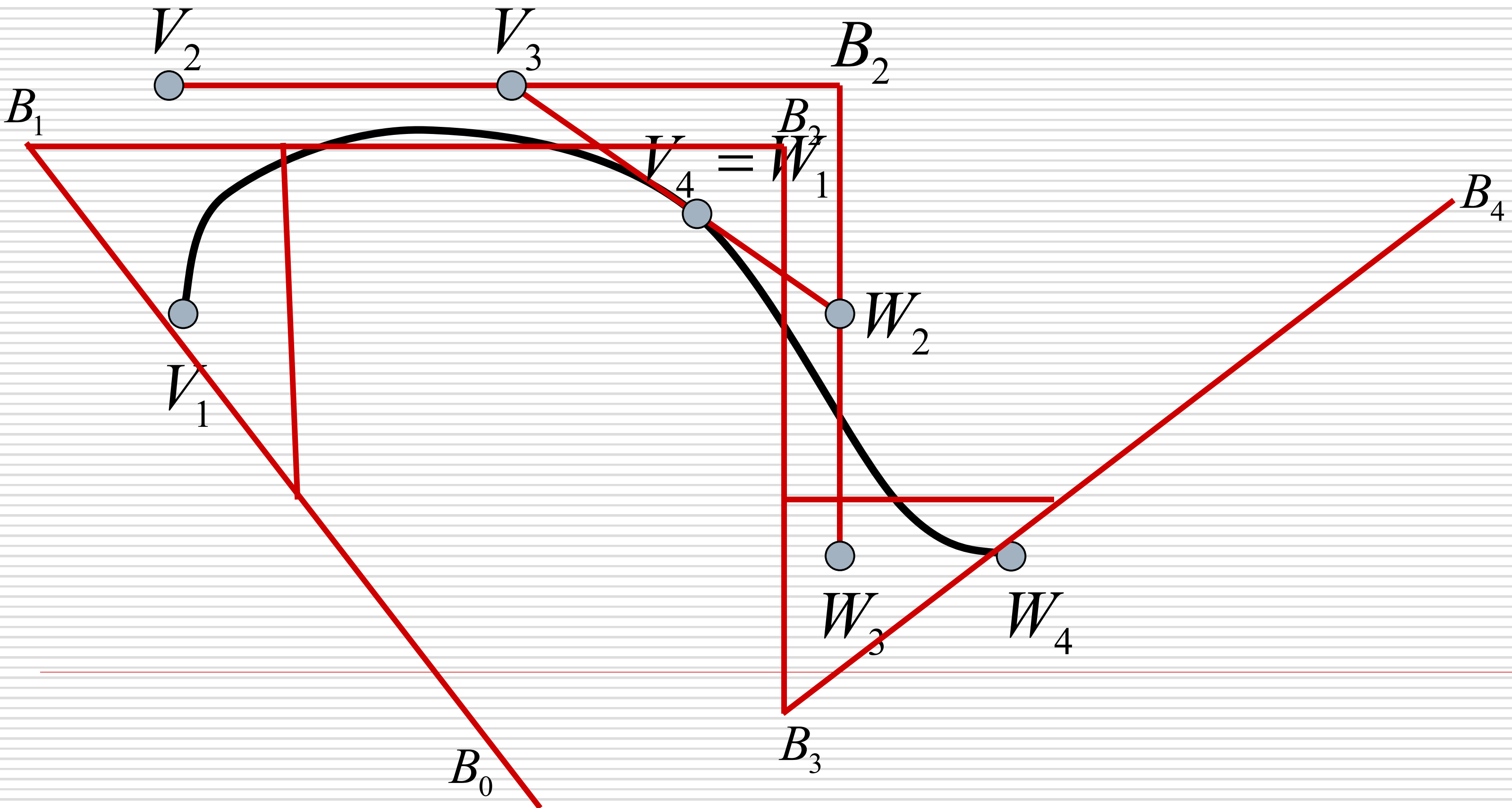

B-Spline

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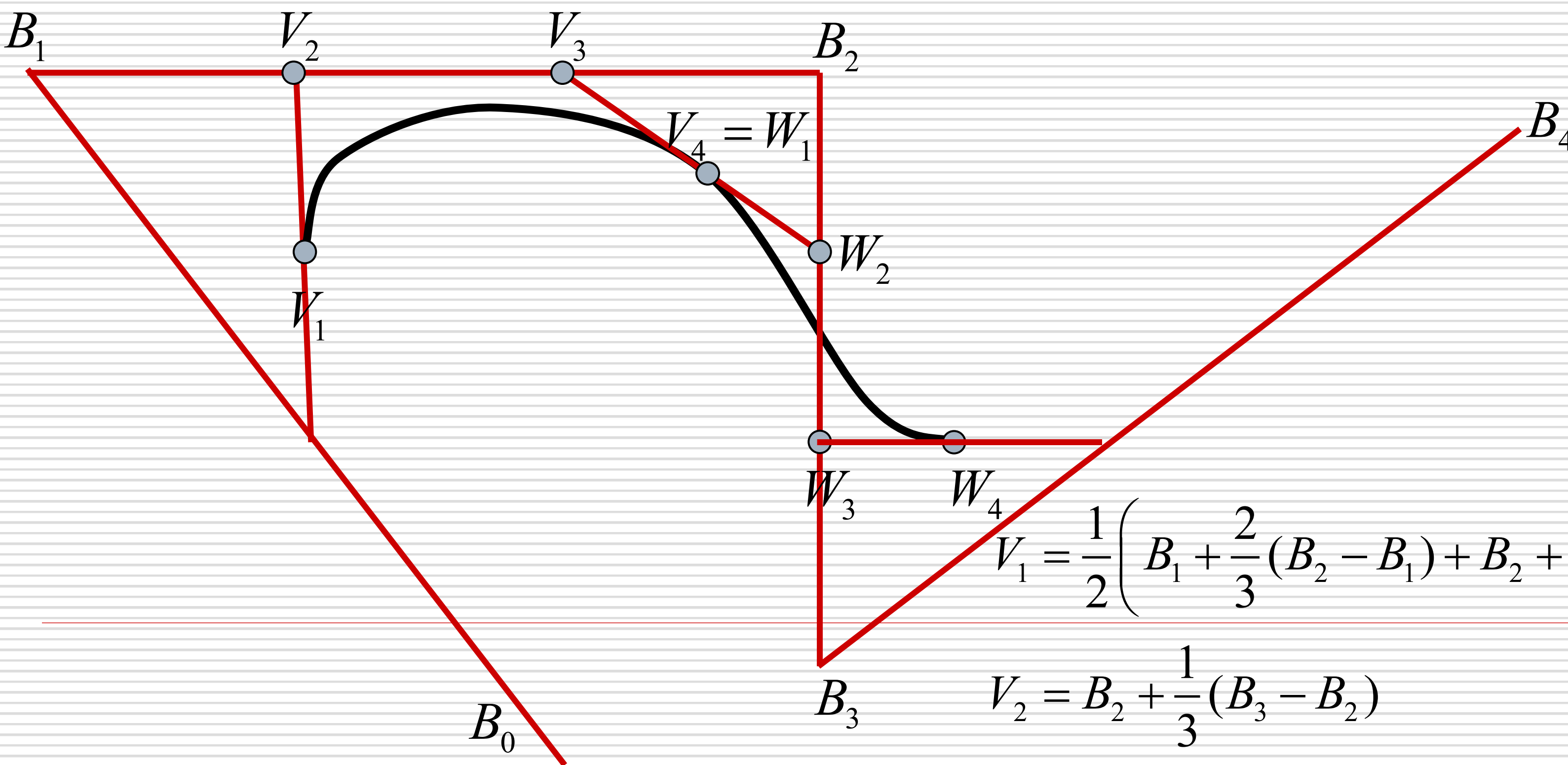
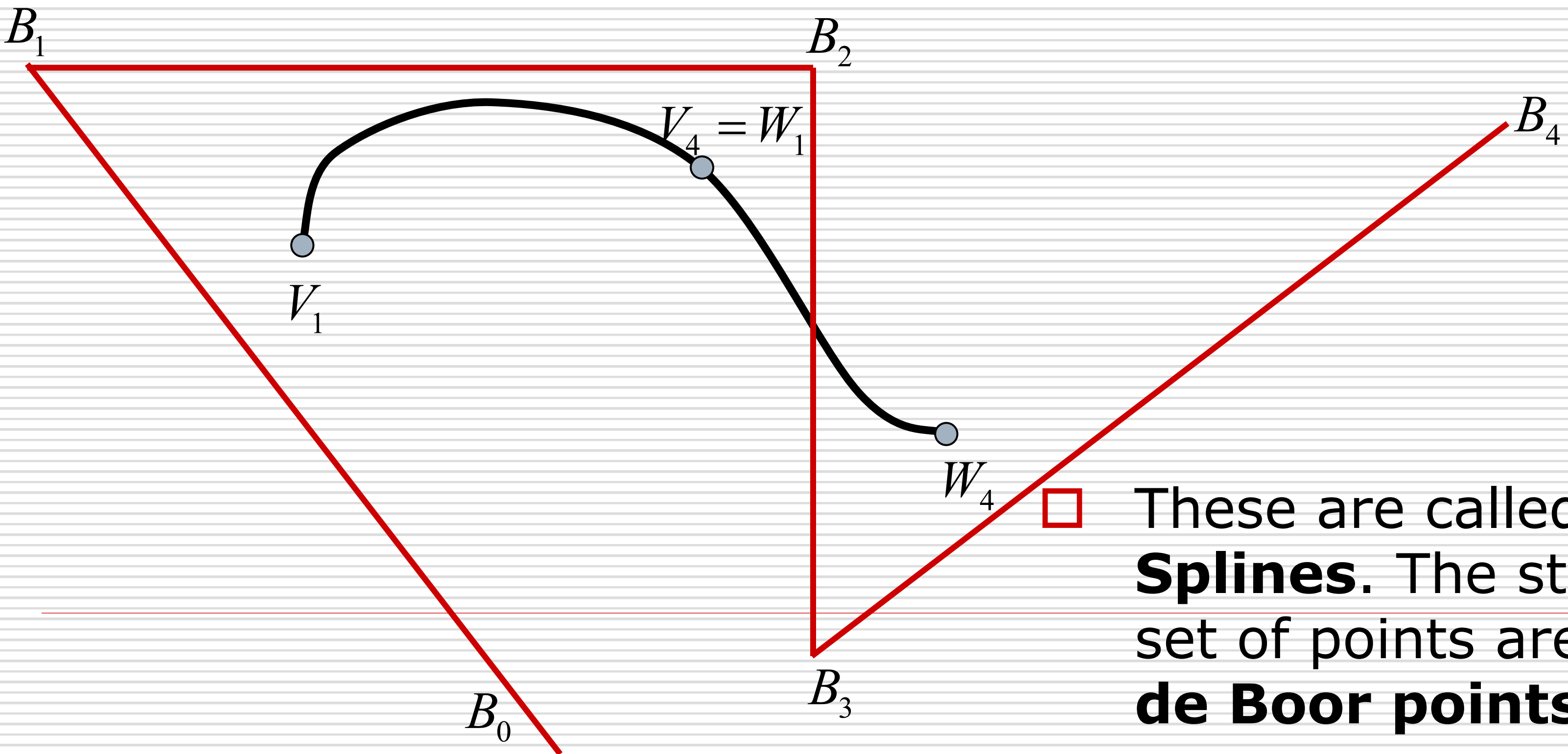


$$V_1 = \frac{1}{2}\left(B_1 + \frac{2}{3}(B_2 - B_1) + B_2 + \frac{1}{3}(B_3 - B_2)\right)$$

$$V_2 = B_2 + \frac{1}{3}(B_3 - B_2)$$

B-Spline

- Instead of specifying the Bézier control points themselves, let's specify the corners of the A-frames in order to build a C^2 continuous spline.



- These are called **B-Splines**. The starting set of points are called **de Boor points**.

Uniform NonRational B-Splines

- cubic B-Spline
  - has $m+1$ control points $P_0, P_1, \ldots, P_m, m \geq 3$
  - has $m-2$ cubic polynomial curve segments $Q_3, Q_4, \ldots, Q_m$
- uniform
  - the knots are spaced at equal intervals of the parameter t
- non-rational
  - not rational cubic polynomial curves

Uniform NonRational B-Splines

- curve segment Q_i is defined by points $P_{i-3}, P_{i-2}, P_{i-1}, P_i$, thus
- **B-Spline geometry matrix**

$$G_{\mathrm{Bs}_i} = \begin{bmatrix} P_{i-3} & P_{i-2} & P_{i-1} & P_i \end{bmatrix}, \quad 3 \le i \le m$$

- if $T_i = \begin{bmatrix} (t-t_i)^3 & (t-t_i)^2 & (t-t_i) & 1 \end{bmatrix}^{\mathrm{T}}$
- then $Q_i(t) = G_{\mathrm{Bs}_i} \bullet M_{\mathrm{Bs}} \bullet T_i, \quad t_i \le t \le t_{i+1}$

Uniform NonRational B-Splines

- so **B-Spline basis matrix**

$$M_{\mathrm{Bs}} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- **B-Spline blending functions**

$$B_{\mathrm{Bs}} = \frac{1}{6}\left[(1-\mathrm{t})^3 \quad 3t^3 - 6t^2 + 4 \quad -3t^3 + 3t^2 + 3t + 1 \quad t^3\right]^{\mathrm{T}}, \quad 0 \le t \le 1$$

NonUniform NonRational B-Splines

- the **knot-value sequence** is a nondecreasing sequence
- allow **multiple knot** and the number of identical parameter is the **multiplicity**
  - Ex. (0,0,0,0,1,1,2,3,4,4,5,5,5,5)
- so

$$Q_i(t) = P_{i-3} \bullet B_{i-3,4}(t) + P_{i-2} \bullet B_{i-2,4}(t) + P_{i-1} \bullet B_{i-1,4}(t) + P_i \bullet B_{i,4}(t)$$

NonUniform NonRational B-Splines

- where $B_{i,j}(t)$ is *j*th-order blending function for weighting control point P_i

$$B_{i,1}(t) = \begin{cases} 1, & t_i \le t \le t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{i,2}(t) = \frac{t - t_i}{t_{i+1} - t_i} B_{i,1}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B_{i+1,1}(t)$$

$$B_{i,3}(t) = \frac{t - t_i}{t_{i+2} - t_i} B_{i,2}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} B_{i+1,2}(t)$$

$$B_{i,4}(t) = \frac{t - t_i}{t_{i+3} - t_i} B_{i,3}(t) + \frac{t_{i+4} - t}{t_{i+4} - t_{i+1}} B_{i+1,3}(t)$$

Knot Multiplicity & Continuity

- since $Q(t_i)$ is within the convex hull of P_{i-3}, P_{i-2}, and P_{i-1}
- if $t_i = t_{i+1}$, $Q(t_i)$ is within the convex hull of P_{i-3}, P_{i-2}, and P_{i-1} and the convex hull of P_{i-2}, P_{i-1}, and P_i, so it will lie on $\overline{P_{i-2}P_{i-1}}$
- if $t_i = t_{i+1} = t_{i+2}$, $Q(t_i)$ will lie on P_{i-1}
- if $t_i = t_{i+1} = t_{i+2} = t_{i+3}$, $Q(t_i)$ will lie on both P_{i-1} and P_i, and the curve becomes broken

Knot Multiplicity & Continuity

- multiplicity 1 : C^2 continuity
- multiplicity 2 : C^1 continuity
- multiplicity 3 : C^0 continuity
- multiplicity 4 : no continuity

NURBS:
NonUniform Rational B-Splines

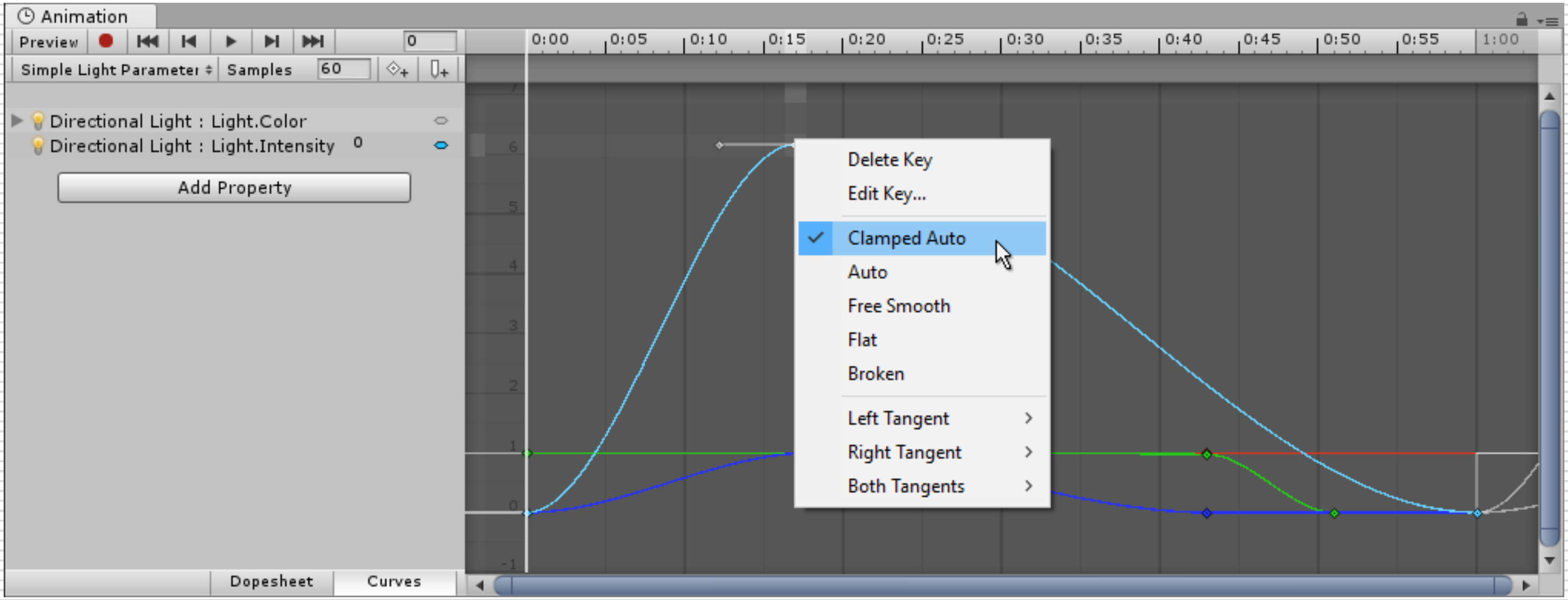
- rational
  - $x(t)$, $y(t)$, and $z(t)$ are defined as the ratio of two cubic polynomials
- rational cubic polynomial curve segments are ratios of polynomials

$$x(t) = \frac{X(t)}{W(t)} \quad y(t) = \frac{Y(t)}{W(t)} \quad z(t) = \frac{Z(t)}{W(t)}$$

- can be Bézier, Hermite, or B-Splines

Parametric Curves in Unity

- No script API supported in standard assets
- AnimationCurve



Parametric Curves in Unity

- Assets store