Discrete Time Signal Processing

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Textbook

Main Textbook

Reference
Activities

- Homework – about three times.
- Tests: twice
  - First test: October 17
  - Second test: to be announced
- Term project
Teach Assistant

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Contents

- Discrete-time signals and systems
- The z-transform
- Sample of continuous-time signals
- Transform analysis of linear time invariant systems
- Structure for discrete-time systems
Contents (continue)

- Filter design techniques
- The discrete Fourier transform
- Computation of the discrete Fourier transform
- Fourier analysis of signals using the discrete Fourier transform
Signals

- Something that conveys information
  - Generally convey information about the state or behavior of a physical system.

- Signal representation
  - represented mathematically as functions of one or more independent variables.
Signal Examples

- Speech signal: represented as a function over time. -- 1D signal
- Image signal: represented as a brightness function of two spatial variables. -- 2D signal
- Ultra sound data or image sequence – 3D signal
Signal Types

- Continuous-time signal
  - defined along a continuum of times and thus are represented by a continuous independent variable.
  - also referred to as analog signal

- Discrete-time signal
  - defined at discrete times, and thus, the independent variable has discrete values:
  - i.e., a discrete-time signal is represented as a sequence of numbers
Signal Types (continue)

- Digital Signals
  - those for which both time and amplitude are discrete

- Signal Processing System: map an input signal to an output signal
  - Continuous-time systems
    - Systems for which both input and output are continuous-time signals
  - Discrete-time system
    - Both input and output are discrete-time signals
  - Digital system
    - Both input and output are digital signals
Example of Discrete-time Signal

- Discrete-time Signal

\[ x = \{x[n]\}, \quad -\infty < n < \infty \]

where \( n \) is an integer
In practice, such sequences can often arise from periodic sampling of an analog signal.

\[ x = x_a[nT], -\infty < n < \infty \]
Signal Operations

- Multiplication and addition
  - The product and sum of two sequences $x[n]$ and $y[n]$ are defined as the sample-by-sample product and sum, respectively.
  - Multiplication by a number $a$ is defined as multiplication of each sample value by $a$.
- Shift operation: $y[n]$ is a delayed or shifted version of $x[n]$

$$y[n] = x[n - n_0]$$

where $n_0$ is an integer.
A Particular Signal

- Unit sample sequence
  - Unit impulse function, Dirac delta function, impulse

\[ \delta[n] = \begin{cases} 
0 & n \neq 0 \\
1 & n = 0 
\end{cases} \]
Signal Representation

An arbitrary sequence can be represented as a sum of scaled, delayed impulses.

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \]
Some Signal Examples

- Unit step sequence

\[ u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases} \]
Some Signal Examples (cont.)

- Real exponential sequence \( x[n] = A \alpha^n \)

\[
y[n] = \begin{cases} 
  A \alpha^n & n \geq 0 \\
  0 & n < 0 
\end{cases}
\]

- \( y[n] \) can be represented as \( y[n] = A \alpha^n u[n] \)
Some Signal Examples (cont.)

- Sinusoidal sequence

\[ x[n] = A \cos(w_0 n + \phi) \]
Consider an exponential sequence \( x[n] = A \alpha^n \), where \( \alpha \) is a complex number having real and imaginary parts

\[
x[n] = A \alpha^n = |A| e^{j\phi} |\alpha|^n e^{j(w_0 n)}
\]

\[
= |A| |\alpha|^n e^{j(w_0 n + \phi)} = |A\alpha|^n (\cos(w_0 n + \phi) + j \sin(w_0 n + \phi))
\]

The sequence oscillates with an exponentially growing envelope if \( |\alpha| > 1 \), or with an exponentially decaying envelope if \( |\alpha| < 1 \)
Complex Exponential Sequence (cont.)

- If $\alpha = 1$, the resulted sequence is referred to as a complex exponential sequence and has the form
  \[ x[n] = |A|e^{j(w_0 n + \phi)} = |A|\left(\cos(w_0 n + \phi) + j \sin(w_0 n + \phi)\right) \]

- The real and imaginary parts of $e^{j(w_0 n + \phi)}$ vary sinusoidally with $n$.
- $w_0$ is called the frequency of the complex exponential and $\phi$ is called the phase.
Discrete and Continuous-time Complex Exponential: Differences

- We only need to consider frequencies in an interval of length $2\pi$, such as $-\pi < w_0 \leq \pi$, or $0 \leq w_0 < 2\pi$. Since
  \[
x[n] = |Ae^{j(w_0n + \phi)} = |Ae^{j(w_0n + \phi + 2\pi n)} = |Ae^{j((w_0 + 2\pi) n + \phi)}\]

- This property holds also for discrete sinusoidal signals: ($r$ is an integer)
  \[
x[n] = A \cos(w_0 n + \phi) = A \cos((w_0 + 2\pi r) n + \phi)\]

- This property does not hold for continuous-time complex exponential signals.
In a continuous-time signal, both complex exponentials and sinusoids are periodic: the period is equal to $2\pi$ divided by the frequency.

In the discrete-time case, a periodic sequence shall satisfy $x[n] = x[n+N]$, for all $n$.

$$e^{jw_0(n+N)} = e^{jw_0n}$$

So, if a discrete-time complex exponential is periodical, then $w_0N = 2\pi k$ shall be hold.
Example

- Consider a signal $x_1[n] = \cos(\pi n/4)$, the signal has a period of $N=8$.
- Let $x_2[n] = \cos(3\pi n/8)$, which has a higher frequency than $x_1[n]$ but $x_2[n]$ is not periodic with period 8, but has a period of $N=16$.
  - Contrary to our intuition from continuous-time sinusoids, increasing the frequency of a discrete-time sinusoid does not necessarily decrease the period of the signal.
- Denote $x_3[n] = \cos(n)$, there exists no integer $N$ satisfying that $x_3[n+N] = x_3[n]$. 
Example (continue)

- As $w_0$ increases from zero toward $\pi$ (parts a-d) the sequence oscillates more rapidly.
- As $w_0$ increases from $\pi$ toward $2\pi$ (parts d-a), the sequence oscillation become slower.

(a) $w_0 = 0$ or $2\pi$

(b) $w_0 = \pi/8$ or $15\pi/8$

(c) $w_0 = \pi/4$ or $7\pi/4$

(d) $w_0 = \pi$
Discrete-time Systems

A transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with value $y[n]$.

$$y[n] = T\{x[n]\}$$

$$x[n] \rightarrow T\{\cdot\} \rightarrow y[n]$$
System Examples

- **Ideal Delay**
  - \( y[n] = x[n-n_d] \), where \( n_d \) is a fixed positive integer called the delay of the system.

- **Moving Average**
  - \[ y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \]

- **Memoryless Systems**
  - The output \( y[n] \) at every value of \( n \) depends only on the input \( x[n] \), at the same value of \( n \).
  - Eg. \( y[n] = (x[n])^2 \), for each value of \( n \).
System Examples (continue)

- **Linear System**: If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs. The system is linear if and only if
  - $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$.
  - $T\{ax[n]\} = aT\{x[n]\} = ay[n]$, for arbitrary constant $a$.
  - So, if $x[n] = \sum_k a_k x_k[n]$, $y[n] = \sum_k a_k y_k[n]$ (superposition principle)

- **Accumulator System**

  $$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \text{(is a linear system)}$$
System Examples (continue)

- **Nonlinear System.**
  - Eg. \( w[n] = \log_{10}(|x[n]|) \) is not linear.

- **Time-invariant System:**
  - If \( y[n] = T\{x[n]\} \), then \( y[n-n_0] = T\{x[n-n_0]\} \)
  - The accumulator is a time-invariant system.

- The compressor system (not time-invariant)
  - \( y[n] = x[Mn], \ -\infty < n < \infty \).
System Examples (continue)

- **Causality**
  - A system is causal if, for every choice of $n_0$, the output sequence value at the index $n = n_0$ depends only the input sequence values for $n \leq n_0$.
  - That is, if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$.

- **Eg. Forward-difference system (non causal)**
  - $y[n] = x[n+1] - x[n]$ (The current value of the output depends on a future value of the input)

- **Eg. Background-difference (causal)**
  - $y[n] = x[n] - x[n-1]$
System Examples (continue)

- Stability
  - Bounded input, bounded output (BIBO): If the input is bounded, $|x[n]| \leq B_x < \infty$ for all $n$, then the output is also bounded, i.e., there exists a positive value $B_y$ s.t. $|y[n]| \leq B_y < \infty$ for all $n$.
  
  - Eg., the system $y[n] = (x[n])^2$ is stable.
  
  - Eg., the accumulated system is unstable, which can be easily verified by setting $x[n] = u[n]$, the unit step signal.
A system that is both linear and time invariant is called a linear time invariant (LTI) system.

By setting the input \( x[n] \) as \( \delta[n] \), the impulse function, the output \( h[n] \) of an LTI system is called the impulse response of this system.

- Time invariant: when the input is \( \delta[n-k] \), the output is \( h[n-k] \).
- Remember that the \( x[n] \) can be represented as a linear combination of delayed impulses

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]
\]
Hence

\[ y[n] = T \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \right\} = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\} \]

\[ = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]

Therefore, a LTI system is completely characterized by its impulse response \( h[n] \).
The above operation is called **convolution**, and can be written in short by $y[n] = x[n] * h[n]$.

In a LTI system, the input sample at $n = k$, represented as $x[k] \delta[n-k]$, is transformed by the system into an output sequence $x[k]h[n-k]$ for $-\infty < n < \infty$. 

**Linear Time Invariant Systems** (continue)

- Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
An Illustration Example

\[ x_{-2}[n] = x[-2] \delta[n + 2] \]

\[ y_{-2}[n] = x[-2] h[n + 2] \]
An Illustration Example (continue)

\[ x_0[n] = x[0] \delta[n] \]

\[ y_0[n] = x[0] h[n] \]

\[ x_3[n] = x[3] \delta[n - 3] \]

\[ y_3[n] = x[3] h[n - 3] \]
An Illustration Example (continue)

\[ x[n] = x_{-2}[n] + x_0[n] + x_3[n] \]

\[ y[n] = y_{-2}[n] + y_0[n] + y_3[n] \]
Convolution can be realized by

- Reflecting $h[k]$ about the origin to obtain $h[-k]$.
- Shifting the origin of the reflected sequences to $k=n$.
- Computing the weighted moving average of $x[k]$ by using the weights given by $h[n-k]$.

Eg.,

- $x[n] = 0, 0, 5, 2, 3, 0, 0…$  
  =$0, 0, 5, 2, 3, 0, 0,…$
- $h[n] = 0, 0, 1, 4, 3, 0, 0…$  
  =$0, 0, 5, 2, 3, 0, 0, 0$
- $x[n] * h[n]$:  
  $0, 0, 0, 20, 8, 12, 0, 0$
  $0, 0, 0, 0, 15, 6, 9, 0$
  $0, 0, 5, 22, 26, 18, 9, 0$
Property of LTI System and Convolution

- **Commutitive**
  - \( x[n] * h[n] = h[n] * x[n] \).

- **Distributive over addition**
  - \( x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] \).

- **Cascade connection**
Property of LTI System and Convolution (continue)

- Parallel combination of LTI systems and its equivalent system.
Stability: A LTI system is stable if and only if

\[ S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty \]

Since

\[ |y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]| < \infty \]

when \(|x[n]| \leq B_x\).

This is a sufficient condition proof.
Stability: necessary condition

- to show that if $S = \infty$, then the system is not BIBO stable, i.e., there exists a bounded input that causes unbounded output.

- Such a bounded input can be set as ($h^*[n]$ is the complex conjugate of $h[n]$)

$$x[n] = \begin{cases} 
        h^*[-n]/|h[-n]|, & h[n] \neq 0 \\
        0, & h[n] = 0
\end{cases}$$

- In this case, the value of the output at $n = 0$ is

$$y[0] = \sum_{-\infty}^{\infty} x[-k]h[k] = \sum_{-\infty}^{\infty} |h[k]|^2 / |h[k]| = S = \infty$$
Property of LTI System and Convolution (continue)

- **Causality**
  - those systems for which the output depends only on the input samples $y[n_0]$ depends only the input sequence values for $n \leq n_0$.
  - Follow this property, an LTI system is causal iff
    $$h[n] = 0 \quad \text{for all } n < 0.$$ 
  - Causal sequence: a sequence that is zero for $n < 0$. A causal sequence could be the impulse response of a causal system.
Impulse Responses of Some LTI Systems

- Ideal delay: \( h[n] = \delta[n-n_d] \)

- Moving average
  \[
  h[n] = \begin{cases} 
  \frac{1}{M_1 + M_2 + 1} & -M_1 \leq n \leq M_2 \\
  0 & \text{otherwise}
  \end{cases}
  \]

- Accumulator
  \[
  h[n] = \begin{cases} 
  1 & n \geq 0 \\
  0 & \text{otherwise}
  \end{cases}
  \]

- Forward difference: \( h[n] = \delta[n+1] - \delta[n] \)

- Backward difference: \( h[n] = \delta[n] - \delta[n-1] \)
In the above, moving average, forward difference and backward difference are stable systems, since the impulse response has only a finite number of terms.

- Such systems are called finite-duration impulse response (FIR) systems.
- FIR is equivalent to a weighted average of a sliding window.
- FIR systems will always be stable.

The accumulator is unstable since \( S = \sum_{n=0}^{\infty} u[n] = \infty \)
Examples of Stable/Unstable Systems (continue)

- When the impulse response is infinite in duration, the system is referred to as an infinite-duration impulse response (IIR) system.
  - The accumulator is an IIR system.

- Another example of IIR system: \( h[n] = a^n u[n] \)
  - When \( |a| < 1 \), this system is stable since
    \[
    S = 1 + |a| + |a|^2 + \ldots + |a|^n + \ldots = 1/(1-|a|) \]
    is bounded.
  - When \( |a| \geq 1 \), this system is unstable
Examples of Causal Systems

- The ideal delay, accumulator, and backward difference systems are causal.
- The forward difference system is noncausal.
- The moving average system is causal requires \(-M_1 \geq 0\) and \(M_2 \geq 0\).
A LTI system can be realized in different ways by separating it into different subsystems.

Equation:

\[
h[n] = (\delta[n+1] - \delta[n]) \ast \delta[n-1] \\
= \delta[n-1] \ast (\delta[n+1] - \delta[n]) \\
= \delta[n] - \delta[n-1]
\]
Another example of cascade systems – inverse system.

\[ h[n] = u[n] \ast (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n] \]