The Fourier transforms of $x_s(t)$ consists of periodically repeated copies of the Fourier transform of $x_c(t)$.

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$, where $\Omega_s = \frac{2\pi}{T}$.
Summary (C/D converter)

- Aliasing effect: If \( \Omega_s \leq 2\Omega_N \), the copies of \( X_c(j\Omega) \) overlap, where \( \Omega_N \) is the highest nonzero frequency component of \( X_c(j\Omega) \). \( \Omega_N \) is referred to as the Nyquist frequency.
Summary (C/D converter)

- Ideal C/D converter
  - Relationship between continuous and discrete Fourier transforms

\[ X_s(j\Omega) = X(e^{j\omega}) \bigg|_{\omega=\Omega T} = X(e^{j\Omega_T}) \]

Because, from another point of view, \( X_s(j\Omega) \) can be represented as the linear combination of a serious of complex exponentials:

\[ X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn} \]

since \( x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \),

\[ = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \bigg|_{\omega=\Omega T} \]
So, we also have

\[
X(e^{jw}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j \left( \frac{w}{T} - \frac{2\pi k}{T} \right) \right)
\]
Ideal reconstruction filter from uniform samples:

An ideal low-pass filter $H_r(j\Omega)$ that has a cut-off frequency $\Omega_c = \Omega_s/2 = \pi/T$ and gain $T$.

Frequency domain:

$$X_r(j\Omega) = H_r(j\Omega)X(e^{j\Omega T}) = \begin{cases} T, & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

The corresponding impulse response is a sinc function, and the reconstructed signal is

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$
If $X_c(j\Omega)$ is band limited: $X_c(j\Omega) = 0$ for $|\Omega| > \pi/T$,

$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega),$$

where

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & \Omega < \pi/T \\ 0, & \Omega \geq \pi/T \end{cases}$$
Summary (Discrete-time processing of continuous-time signals)

In time domain: *impulse invariance property*

- $h[n] = T h_c(nT)$, i.e., the impulse response of the discrete-time system is a scaled, sampled version of $h_c(t)$. 
Summary (Continuous-time processing of discrete-time signals)

\[ H_{\text{eff}}(e^{jw}) = H_c\left( j \frac{w}{T} \right), \quad |w| < \pi \]
Summary (Changing the Sampling rate using discrete-time processing)

- Compressor; down-sampling
- Since this is a ‘re-sampling’ process. Similar to D/C converter, the frequency domain relationship:

\[
X_d(e^{jw}) = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(w-2\pi i)/M})
\]
Summary (Changing the Sampling rate using discrete-time processing)

- From the above, the DFT of the down-sampled signal is the superposition of $M$ shifted/scaled versions of the DFT of the original signal.

- To avoid aliasing, we need $w_N < \pi/M$, where $w_N$ is the highest frequency of the discrete-time signal $x[n]$.

Decimator: a low-pass filter followed by down-sampling
Summary (Changing the Sampling rate using discrete-time processing)

\[
\begin{align*}
\uparrow L & & \uparrow L & & \uparrow L \\
x[n] & & x_e[n] & & x_t[n] \\
\text{Sampling period } T & & \text{Sampling period } T' = T/L & & \text{Sampling period } T' = T/L
\end{align*}
\]

\[
x_e[n] = \begin{cases} 
x[n / L], & n = 0, \pm L, \pm 2L, \ldots \\
0, & \text{otherwise}
\end{cases}
\]

- Equivalently, \( x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \)
- Expander; up-sampling
- Frequency-domain relationship \( X_e(e^{jw}) = X(e^{jwL}) \)
Summary (Changing the Sampling rate using discrete-time processing)

- Interpolator: a low-pass filter followed by up-sampling
- Similar to the ideal D/C converter,
  - If we choose an ideal lowpass filter with cutoff frequency $\pi/L$ and gain $L$, its impulse response is
    $$h_i[n] = \frac{\sin(\pi n/L)}{\pi n / L}$$
  - Hence
    $$x_i[n] = x_e[n] * h_i[n] = \left( \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] \right) * h_i[n]$$
    $$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}$$
Summary (Changing the Sampling rate using discrete-time processing)

- Changing the sampling rate by a non-integer factor
  - $T' = T M / L$
Summary (Changing the Sampling rate using discrete-time processing)