DTFT continue (c.f. Shenoi, 2006)

- We have introduced DTFT and showed some of its properties. We will investigate them in more detail by showing the associated derivations later.
- We have also given a motivation of DFT which is both discrete in time and frequency domains. We will also introduce DFT in more detail below.

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT} \qquad x(nT) = \frac{T}{2\pi}\int_{-(\pi/T)}^{\pi/T} X(e^{j\omega T})e^{j\omega nT}\,d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

DC response

- When w=0, the complex exponential e^{-jw} becomes a constant signal, and the frequency response $X(e^{jw})$ is often called the DC response when w=0.
  - The term DC stands for direct current, which is a constant current.

We will represent the spectrum of DTFT either by $H(e^{jwT})$ or more often by $H(e^{jw})$ for convenience.

- When represented as $H(e^{jw})$, it has the frequency range $[-\pi,\pi]$. In this case, the frequency variable is to be understood as the normalized frequency. The range $[0, \pi]$ corresponds to $[0, w_s/2]$ (where $w_s T=2\pi$), and the normalized frequency π corresponds to the Nyquist frequency (and 2π corresponds to the sampling frequency).

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT} \qquad x(nT) = \frac{T}{2\pi}\int_{-(\pi/T)}^{\pi/T} X(e^{j\omega T})e^{j\omega nT}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DC response

- When w=0, the complex exponential e^{-jw} becomes a constant signal, and the frequency response $X(e^{jw})$ is often called the DC response when w=0.
  - The term DC stands for direct current, which is a constant current.

DTFT Properties Revisited

- **Time shifting**

If $x(n)$ has a DTFT $X(e^{j\omega})$, then $x(n-k)$ has a DTFT equal to $e^{-j\omega k}X(e^{j\omega})$, where k is an integer. This is known as the *time-shifting property* and it is easily proved as follows: DTFT of $x(n-k) = \sum_{n=-\infty}^{\infty} x(n-k)e^{-j\omega n} = e^{-j\omega k}\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = e^{-j\omega k}X(e^{j\omega})$. So we denote this property by

$$x(n-k) \Leftrightarrow e^{-j\omega k}X(e^{j\omega})$$

➢ Frequency shifting

If $x(n) \Leftrightarrow X(e^{j\omega})$, then

$$e^{j\omega_0 n} x(n) \Leftrightarrow X(e^{j(\omega-\omega_0)})$$

This is known as the *frequency-shifting property*, and it is easily proved as follows:

$$\sum_{n=-\infty}^{\infty} x(n) e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)})$$

➢Time reversal

$$\text{if } x(n) \Leftrightarrow X(e^{j\omega}) \text{ then}$$
$$x(-n) \Leftrightarrow X(e^{-j\omega})$$

Proof: DTFT of $x(-n) = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n}$. We substitute $(-n) = m$, and we get $\sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{j\omega m} = \sum_{m=-\infty}^{\infty} x(m)e^{-j(-\omega)m} = X(e^{-j\omega})$.

➢ DTFT of δ(n)

$$\sum_{n=-\infty}^{n} \delta(n) e^{-jwn} = e^{jw0} = 1$$

➢DTFT of δ(n+k)+ δ(n-k)

According to the time-shifting property,

$$\text{DTFT of } \delta(n+k) \text{ is } e^{jwk}, \text{ DTFT of } \delta(n-k) \text{ is } e^{-jwk}$$

Hence

$$\text{DTFT of } \delta(n+k)+\delta(n-k) \text{ is } e^{jwk} + e^{-jwk} = 2\cos(wk)$$

➢ DTFT of x(n) = 1 (for all n)

x(n) can be represented as $x(n) = \sum_{k=-\infty}^{\infty} \delta(n-k)$

We prove that its DTFT is $2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

Proof: The inverse DTFT of $2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ is evaluated as

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j\omega n} \, d\omega$$

$$= \int_{-\pi}^{\pi} \left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j\omega n} \, d\omega$$

From the sifting property we get

$$\left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j\omega n} = \left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j2\pi k n}$$

$$= \left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right]$$

where we have used $e^{j2\pi kn} = 1$ for all n. When we integrate the sequence of impulses from $-\pi$ to π, we have only the impulse at $\omega = 0$.

Hence

$$\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)\right]e^{jwn}dw$$

$$=\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)\right]dw$$

$$=\int_{-\pi}^{\pi}\delta(w)dw=\int_{-\infty}^{-\pi}\delta(w)dw+\int_{-\pi}^{\pi}\delta(w)dw+\int_{\pi}^{\infty}\delta(w)dw$$

$$=\int_{-\infty}^{\infty}\delta(w)dw=1 \qquad \text{for all } n$$

From another point of view

➢ According to the sampling property: the DTFT of a continuous signal $x_a(t)$ sampled with period T is obtained by a periodic duplication of the continuous Fourier transform $X_a(jw)$ with a period $2\pi/T = w_s$ and scaled by T.

Since the continuous F.T. of x(t)=1 (for all t) is $\delta(t)$, the DTFT of x(n)=1 shall be a impulse train (or impulse comb), and it turns out to be

$$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

➢ DTFT of $a^n u(n)$ (|a|<1)

let $x_1(n) = a^n u(n)$

then $$X_1(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

This infinite sequence converges to $1/(1 - ae^{-j\omega}) = e^{j\omega}/(e^{j\omega} - a)$ **when |a|<1.**

➢ DTFT of Unit Step Sequence

Note that $a^n u(n) \Leftrightarrow 1/(1 - ae^{-j\omega}) = e^{j\omega}/(e^{j\omega} - a)$ is valid only when $|a| < 1$.
When $a = 1$, we get the unit step sequence $u(n)$

We express the unit step function as the sum of two functions

$$u(n) = u_1(n) + u_2(n)$$

where

$$u_1(n) = \tfrac{1}{2} \quad \text{for} \quad -\infty < n < \infty$$

and

$$u_2(n) = \begin{cases} \tfrac{1}{2} & \text{for } n \geq 0 \\ -\tfrac{1}{2} & \text{for } n < 0 \end{cases}$$

Therefore we express $\delta(n) = u_2(n) - u_2(n-1)$. Using $\delta(n) \Leftrightarrow 1$ and $u_2(n) - u_2(n-1) \Leftrightarrow U_2(e^{j\omega}) - e^{-j\omega}U_2(e^{j\omega}) = U_2(e^{j\omega})[1 - e^{-j\omega}]$, and equating the two results, we get

$$1 = U_2(e^{j\omega})[1 - e^{-j\omega}]$$

Therefore

$$U_2(e^{j\omega}) = \frac{1}{[1 - e^{-j\omega}]}$$

We know that the DTFT of $u_1(n) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) = U_1(e^{j\omega})$.

Adding these two results, we have the final result

$$u(n) \Leftrightarrow \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) + \frac{1}{(1 - e^{-j\omega})}$$

This gives us the DTFT of the unit step function $u(n)$, which is unique.

➢ Differentiation Property

To prove that $nx(n) \Leftrightarrow j[dX(e^{j\omega})]/d\omega$,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

differentiate both sides to get

$$[dX(e^{j\omega})]/d\omega = \sum_{n=-\infty}^{\infty} x(n)(-jn)e^{-j\omega n}$$

multiplying both sides by j, we get

$$j[dX(e^{j\omega})]/d\omega = \sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}.$$

➢ DTFT of a rectangular pulse

Consider a rectangular pulse

$$x_r(n) = \begin{cases} 1 & |n| \leq N \\ 0 & |n| > N \end{cases}$$

Its DTFT is derived as follows:

$$X_r(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n}$$

To simplify this summation, we use the identity[5]

$$\sum_{n=-N}^{N} r^n = \frac{r^{N+1} - r^{-N}}{r - 1}; \quad r \neq 1$$

$$= 2N + 1; \quad r = 1$$

and get

$$X_r(e^{j\omega}) = \frac{e^{-j(N+1)\omega} - e^{jN\omega}}{e^{-j\omega} - 1}$$

$$= \frac{e^{-j0.5\omega}\left(e^{-j(N+0.5)\omega} - e^{j(N+0.5)\omega}\right)}{e^{-j0.5\omega}(e^{-j0.5\omega} - e^{j0.5\omega})}$$

$$= \begin{cases} \dfrac{\sin[(N+0.5)\omega]}{\sin[0.5\omega]} & \omega \neq 0 \\ 2N+1 & \omega = 0 \end{cases}$$

Convolution

Convolution of two discrete-time signals

Let x[n] and h[n] be two signals, the convolution of x and h is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

can be written in short by y[n] = x[n] $*$ h[n].

Convolution of two continuous-time signals

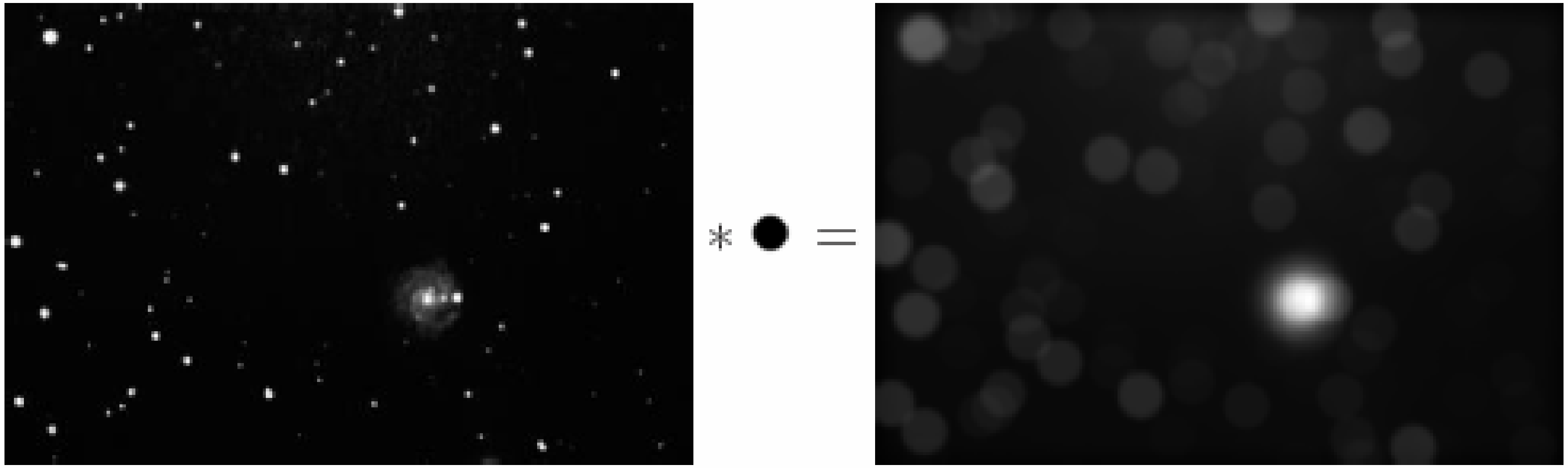
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

can be written in short by y(t) = x(t) $*$ h(t)

From Kuhn 2005

Continuous convolution: optics example

If a projective lens is out of focus, the blurred image is equal to the original image convolved with the aperture shape (e.g., a filled circle):

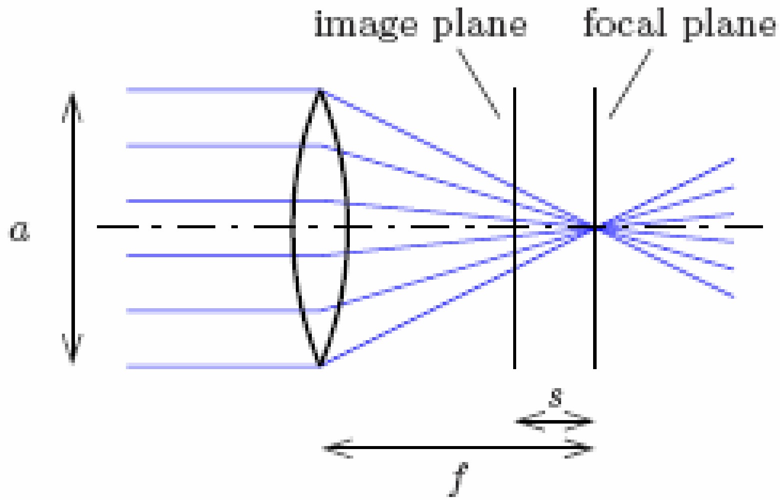


Point-spread function h (disk, $r = \frac{as}{2f}$):

$$h(x,y) = \begin{cases} \frac{1}{r^2\pi}, & x^2+y^2 \le r^2 \\ 0, & x^2+y^2 > r^2 \end{cases}$$

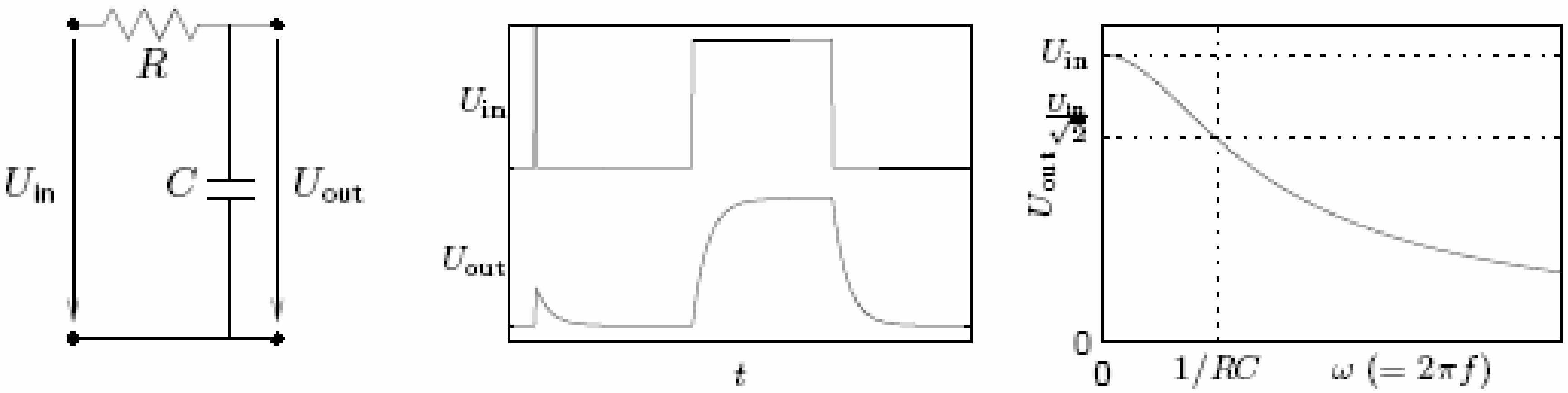
Original image I, blurred image $B = I * h$, i.e.

$$B(x,y) = \iint I(x-x', y-y') \cdot h(x', y') \cdot \mathrm{d}x' \mathrm{d}y'$$



From Kuhn 2005

Continuous convolution: electronics example



Any passive network (R, L, C) convolves its input voltage U_{in} with an *impulse response function* h, leading to $U_{out} = U_{in} * h$, that is

$$U_{out}(t) = \int_{-\infty}^{\infty} U_{in}(t-\tau) \cdot h(\tau) \cdot \mathrm{d}\tau$$

In this example:

$$\frac{U_{in} - U_{out}}{R} = C \cdot \frac{\mathrm{d}U_{out}}{\mathrm{d}t}\,, \qquad h(t) = \begin{cases} \frac{1}{RC} \cdot \mathrm{e}^{\frac{-t}{RC}}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Properties of convolution

- Communitive
  - $x[n] * h[n] = h[n] * x[n]$

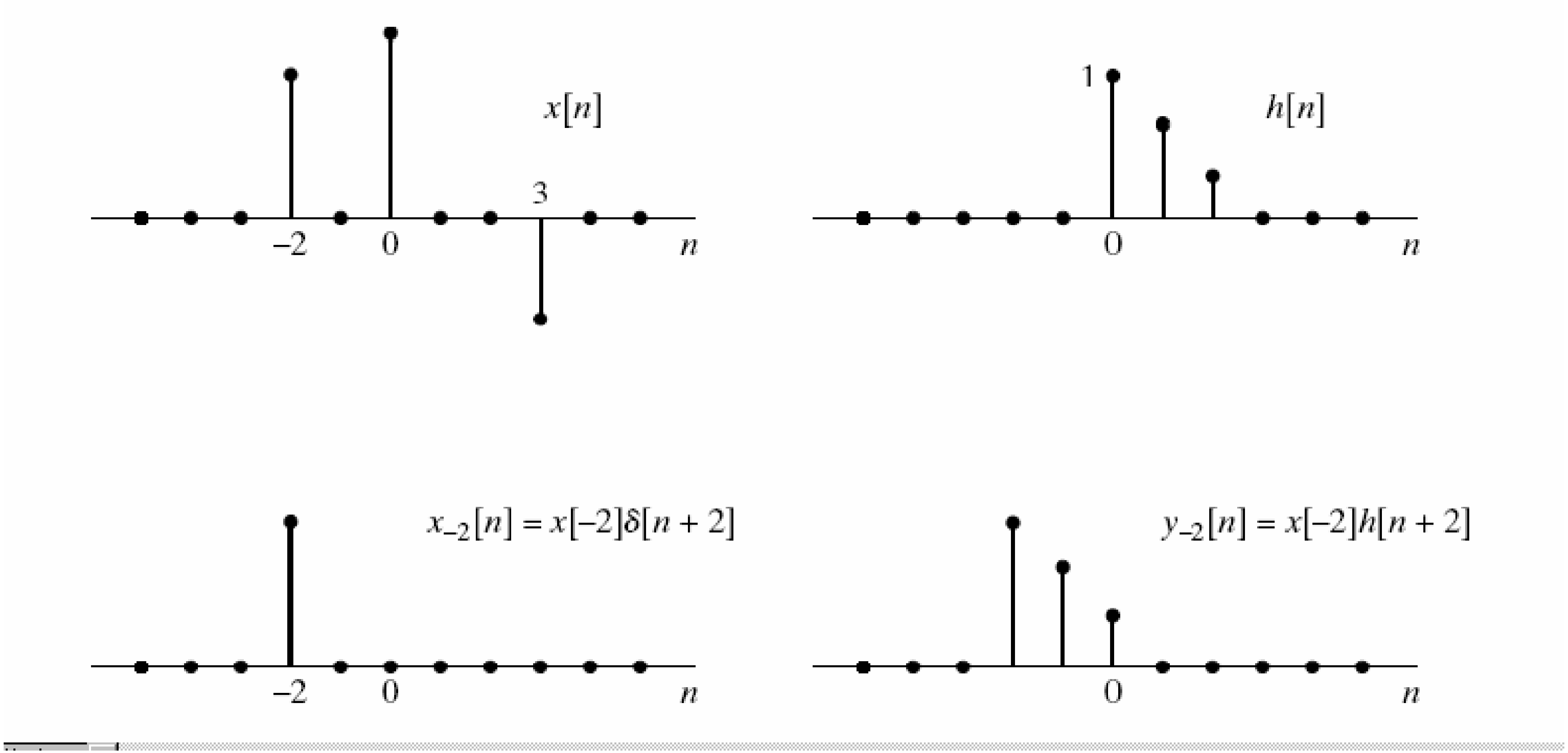
  this means that y[n] can also be represented as

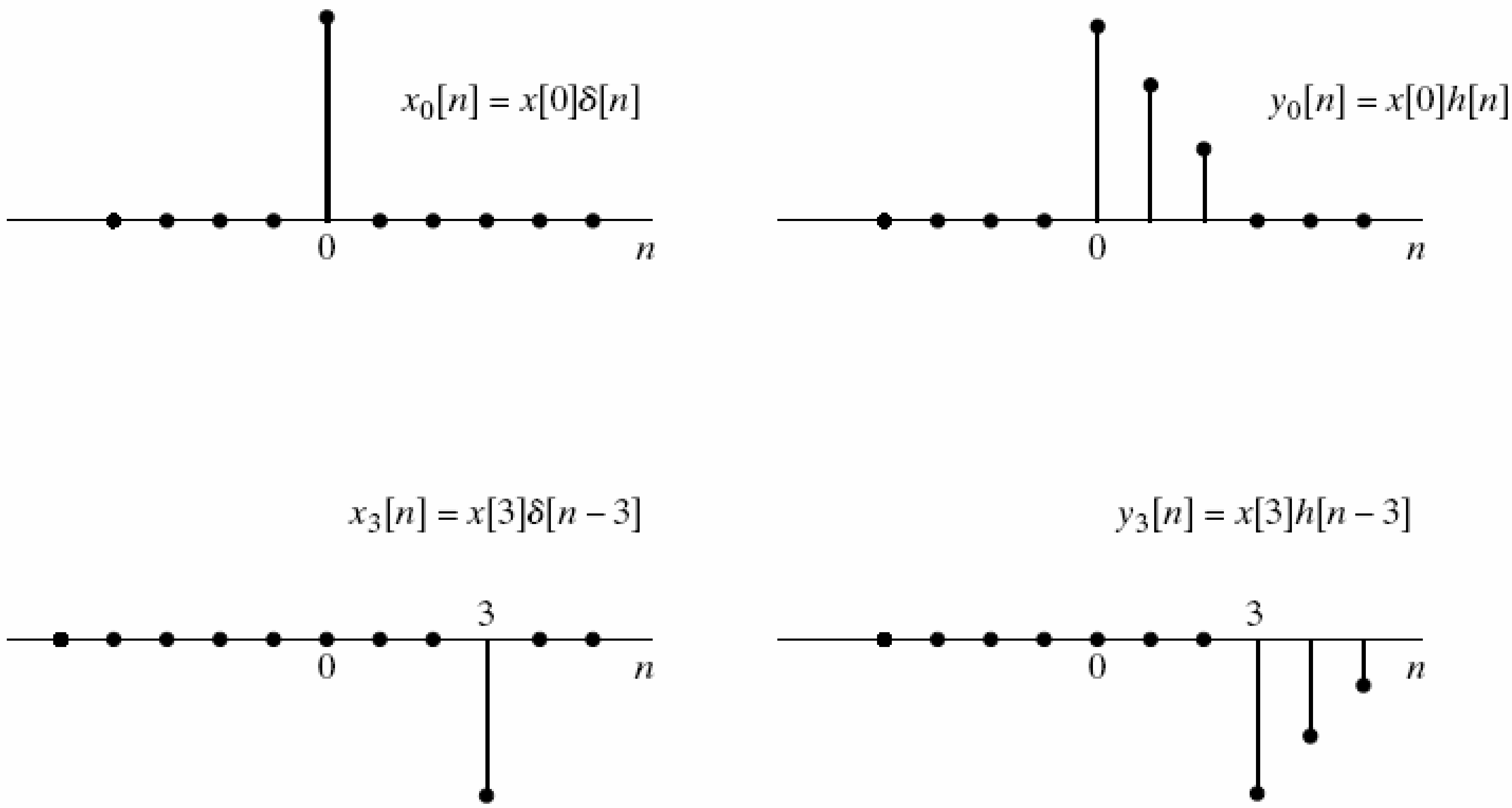
$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

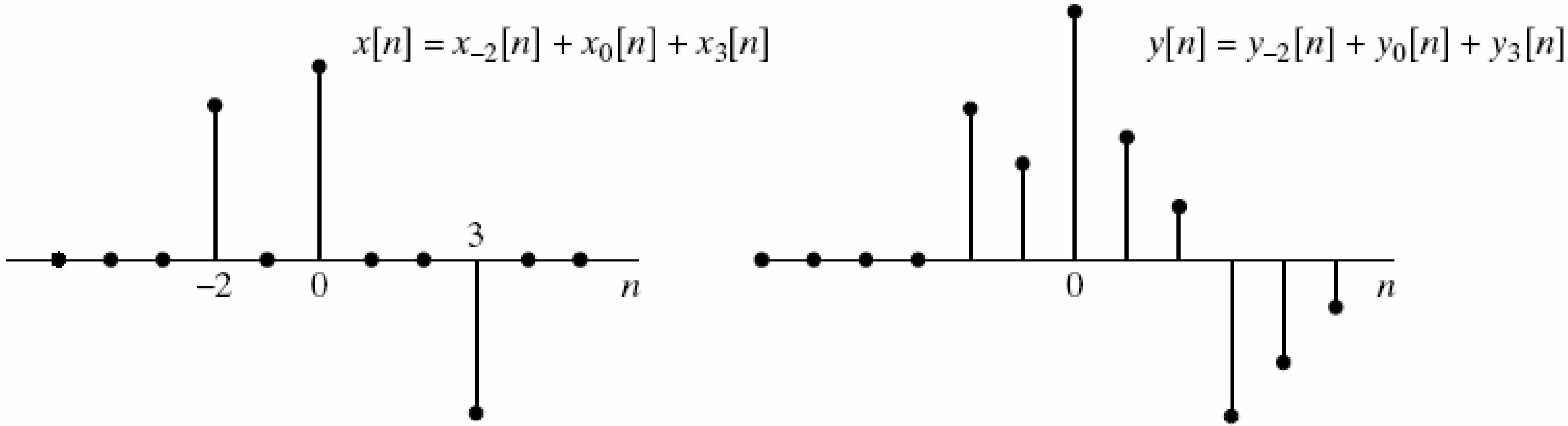
- Associative
  - $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n].$
- Linear
  - $x[n] * (ah_1[n] + bh_2[n]) = ax[n] * h_1[n] + bx[n] * h_2[n].$
- Sequence shifting is equivalent to convolute with a shifted impulse
  - $x[n-d] = x[n] * \delta[n-d]$

An illustrative example

$x[n] * \delta[n]$ $x[n] * h[n]$




$x_0[n] = x[0]\delta[n]$
0
n
$y_0[n] = x[0]h[n]$
0
n
$x_3[n] = x[3]\delta[n-3]$
3
0
n
$y_3[n] = x[3]h[n-3]$
3
0
n



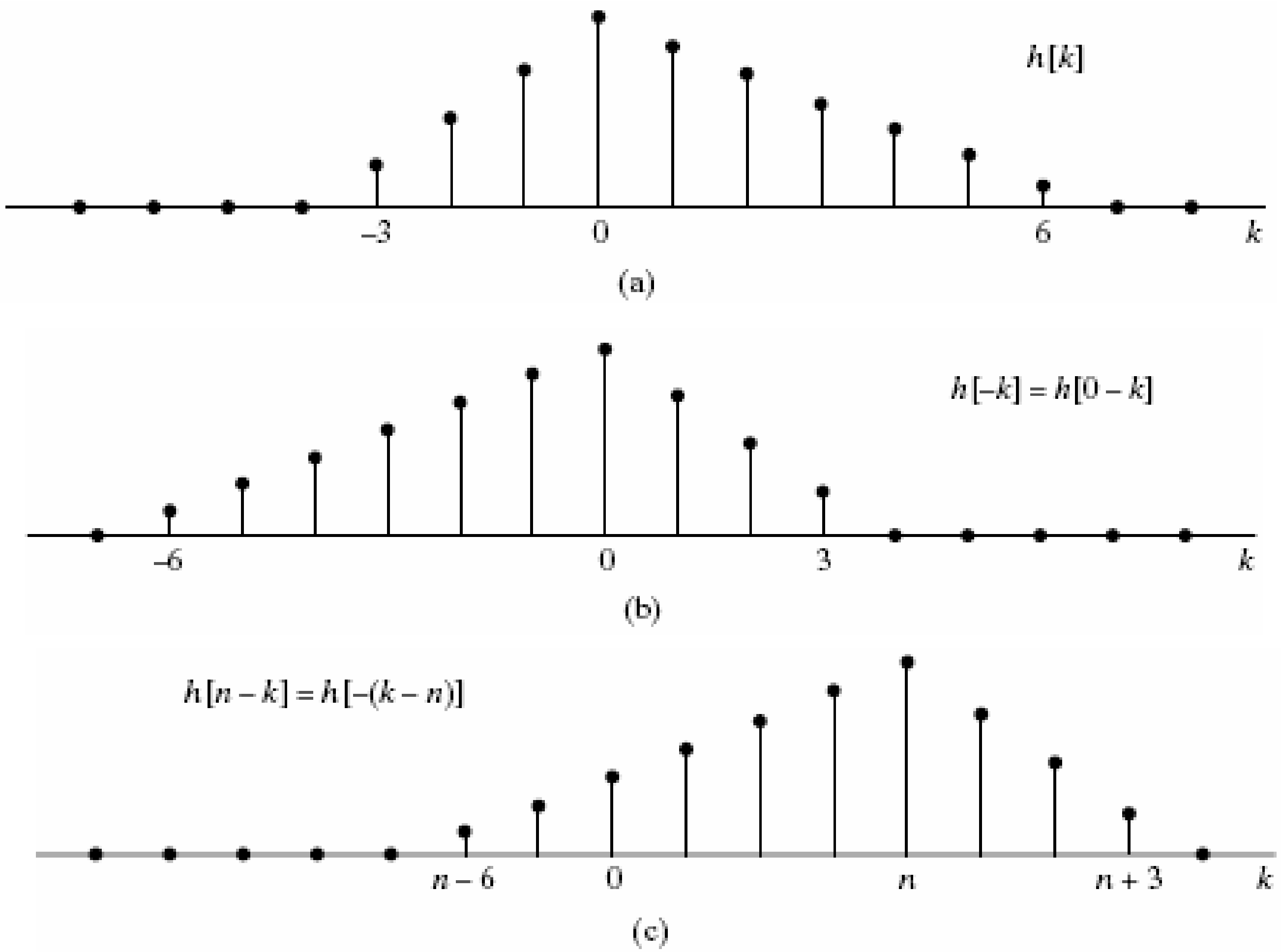
$x[n] = x_{-2}[n] + x_0[n] + x_3[n]$
3
−2
0
n
$y[n] = y_{-2}[n] + y_0[n] + y_3[n]$
0
n

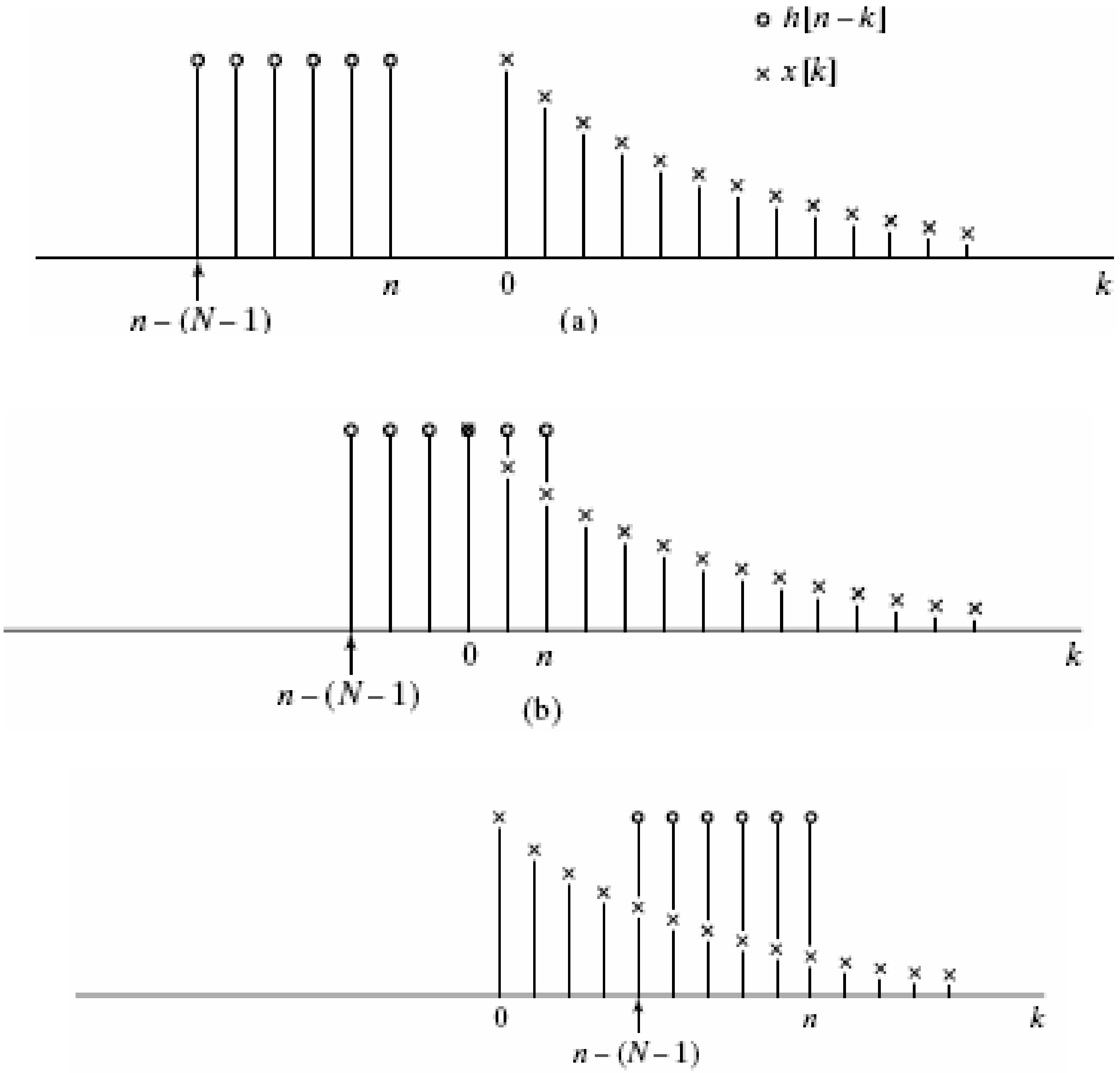
Convolution can be realized by

–Reflecting $h[k]$ about the origin to obtain $h[-k]$.

–Shifting the origin of the reflected sequences to $k=n$.

–Computing the weighted moving average of $x[k]$ by using the weights given by $h[n-k]$.





h[n − k]
x[k]
n
0
k
n − (N − 1)
(a)
0
n
k
n − (N − 1)
(b)
0
n
k
n − (N − 1)

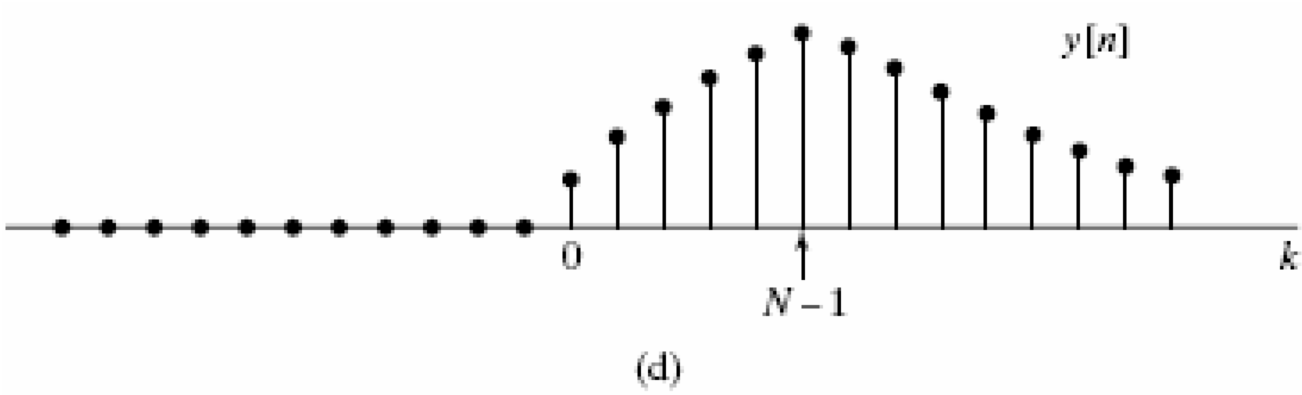


Figure 2.10 Sequence involved in computing a discrete convolution. (a)–(c) The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.) (d) Corresponding output sequence as a function of n.

Convolution can be explained as "arithmetic product."

- Eg.,
  - $x[n]$ = 0, 0, 5, 2, 3, 0, 0...
  - $h[n]$ = 0, 0, 1, 4, 3, 0, 0...
  - $x[n] * h[n]$:

$$
\begin{array}{llllllllll}
 & 0, & 0, & 5, & 2, & 3, & 0, & 0, & \ldots & \\
*) & 0, & 0, & 1, & 4, & 3, & 0, & 0, & \ldots & \\
\hline
 & 0, & 0, & 5, & 2, & 3, & 0, & 0, & 0 & \\
 & 0, & 0, & 0, & 20, & 8, & 12, & 0, & 0 & \\
 & 0, & 0, & 0, & 0, & 15, & 6, & 9, & 0 & \\
\hline
 & 0, & 0, & 5, & 22, & 26, & 18, & 9, & 0 &
\end{array}
$$

Convolution vs. Fourier Transform

Multiplication Property: For continuous F.T. and DTFT, if we perform multiplication in time domain, then it is equivalent to performing convolution in the frequency domain, and vice versa.

DTFT convolution theorem:

Let $x[n] \leftrightarrow X(e^{jw})$ and $h[n] \leftrightarrow H(e^{jw})$.

If $y[n] = x[n] * h[n]$, then

$$Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

modulation/windowing theorem (or multiplication property)

Let $x[n] \leftrightarrow X(e^{jw})$ and $w[n] \leftrightarrow W(e^{jw})$.

If $y[n] = x[n]w[n]$, then

$$Y\left(e^{jw}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) W\left(e^{j(w-\theta)}\right) d\theta$$

a periodic convolution

For continuous F. T.

Continuous form:

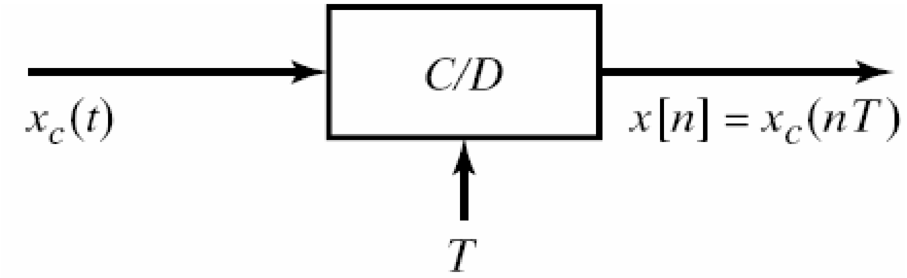
$$\mathcal{F}\{(f * g)(t)\} = \mathcal{F}\{f(t)\} \cdot \mathcal{F}\{g(t)\}$$
$$\mathcal{F}\{f(t) \cdot g(t)\} = \mathcal{F}\{f(t)\} * \mathcal{F}\{g(t)\}$$

In summary,

Convolution in the time domain is equivalent to (complex) scalar multiplication in the frequency domain.

Convolution in the frequency domain corresponds to scalar multiplication in the time domain.

Sampling theorem revisited (oppenheim et al. 1999)



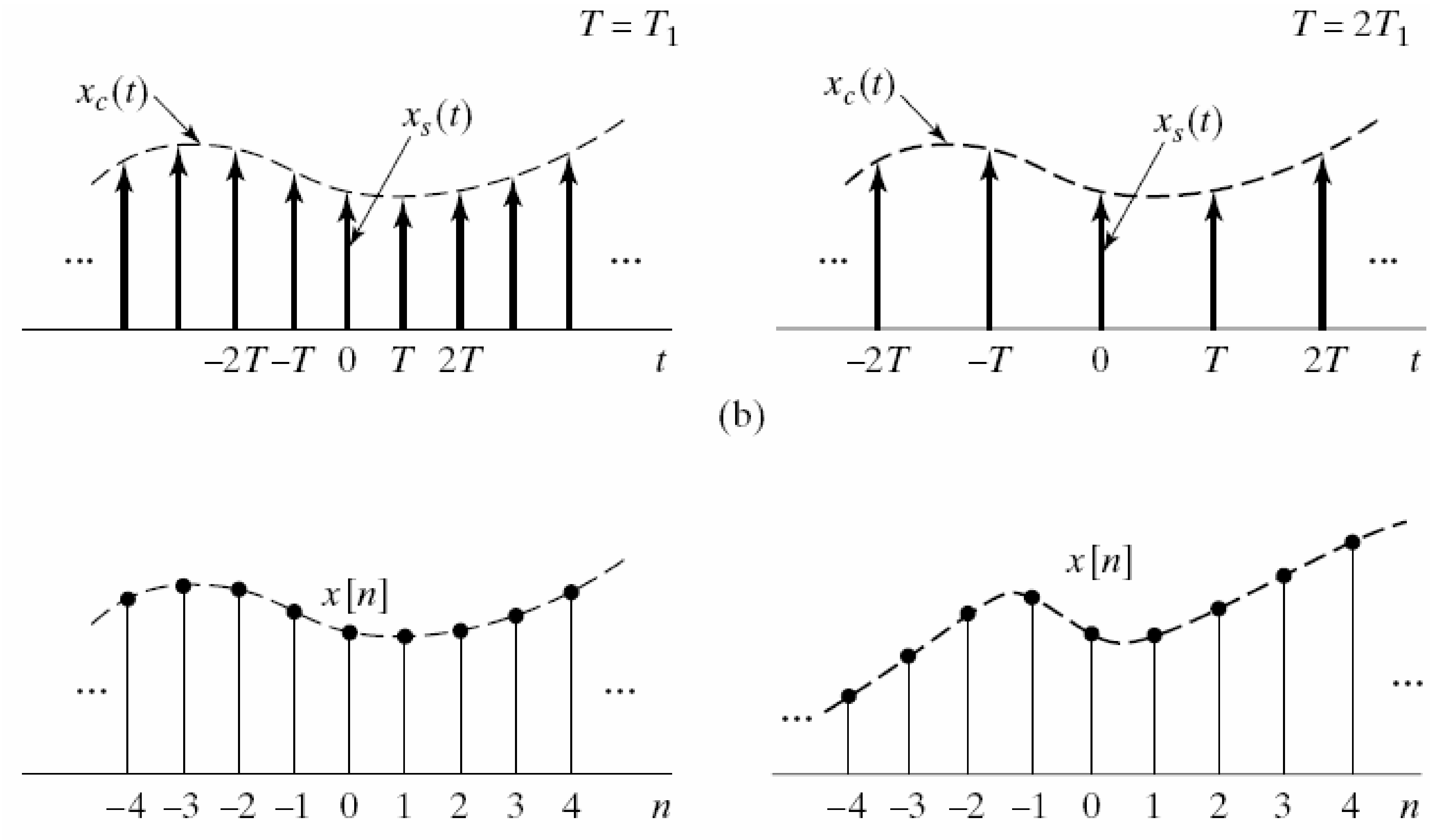
An ideal continuous-to-discrete-time (C/D) converter

Let s(t) be a continuous signal, which is a periodic impulse train:

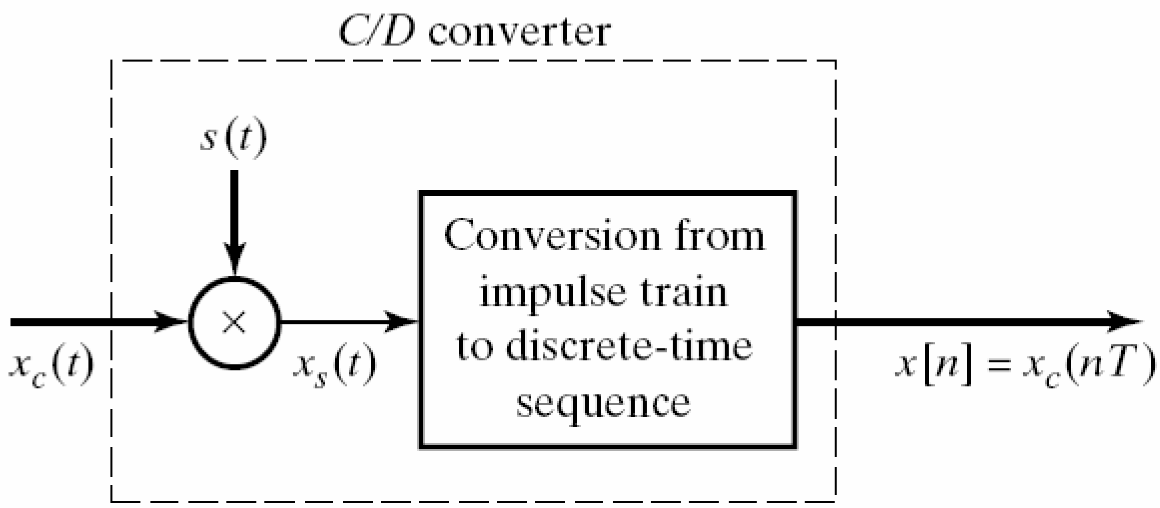
$$s(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

We modulate s(t) with $x_c(t)$, obtaining

$$x_s(t) = x_c(t)s(t) = x_c(t)\sum_{-\infty}^{\infty} \delta(t - nT)$$



Examples of $x_s(t)$ for two sampling rates



Sampling with a periodic impulse followed by conversion to a discrete-time sequence

Through the 'sifting property' of the impulse function, $x_s(t)$ can be expressed as

$$x_s(t) = \sum_{-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

Let us now consider the continuous Fourier transform of $x_s(t)$. Since $x_s(t)$ is a product of $x_c(t)$ and s(t), its continuous Fourier transform is the convolution of $X_c(j\Omega)$ and $S(j\Omega)$.

Note that the continuous Fourier transform of a periodic impulse train is a periodic impulse train.

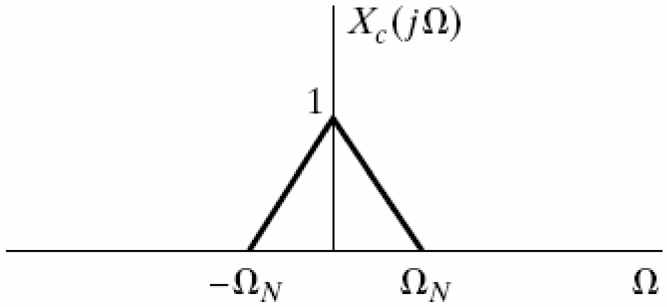
$$S(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

where Ωs =2π/T is the sampling frequency in radians/s.

Since $X_s(j\Omega) = X_c(j\Omega) * S(j\Omega)$

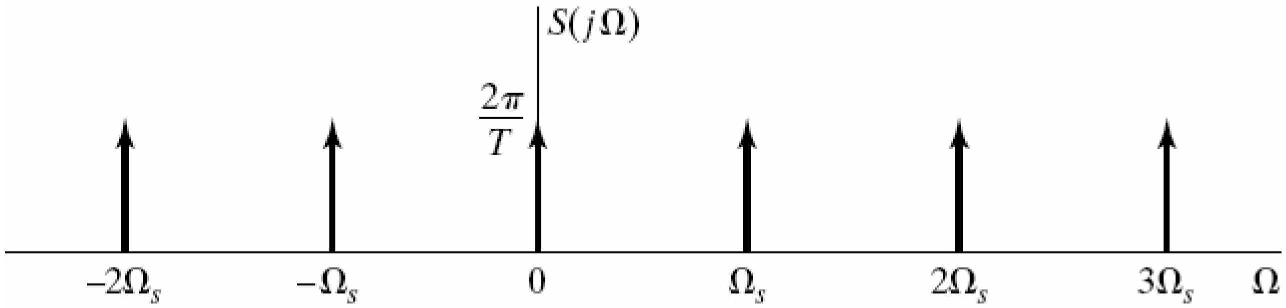
If follows that $X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

Again, we see that the copies of $X_c(j\Omega)$ are shifted by integer multiples of the sampling frequency, and then superimposed to product the periodic Fourier transform of the impulse train of samples.



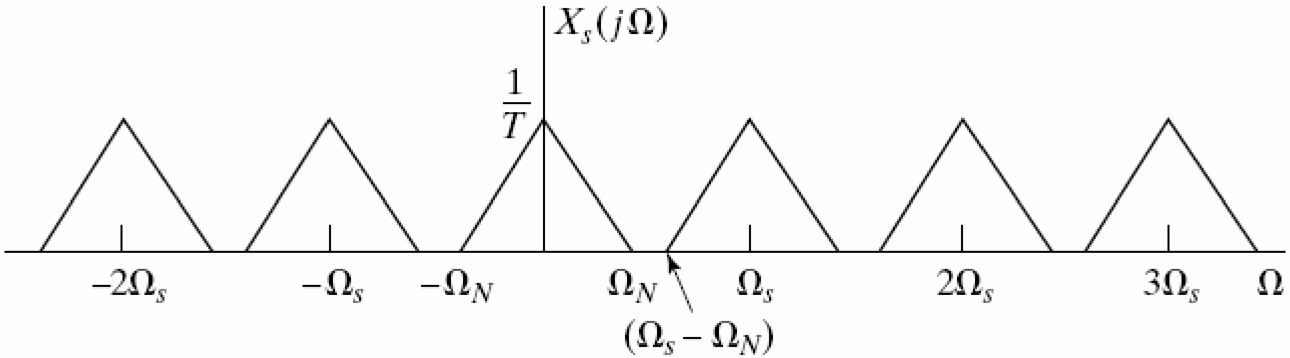
$X_c(j\Omega)$
1
$-\Omega_N$
Ω_N
Ω

(a)



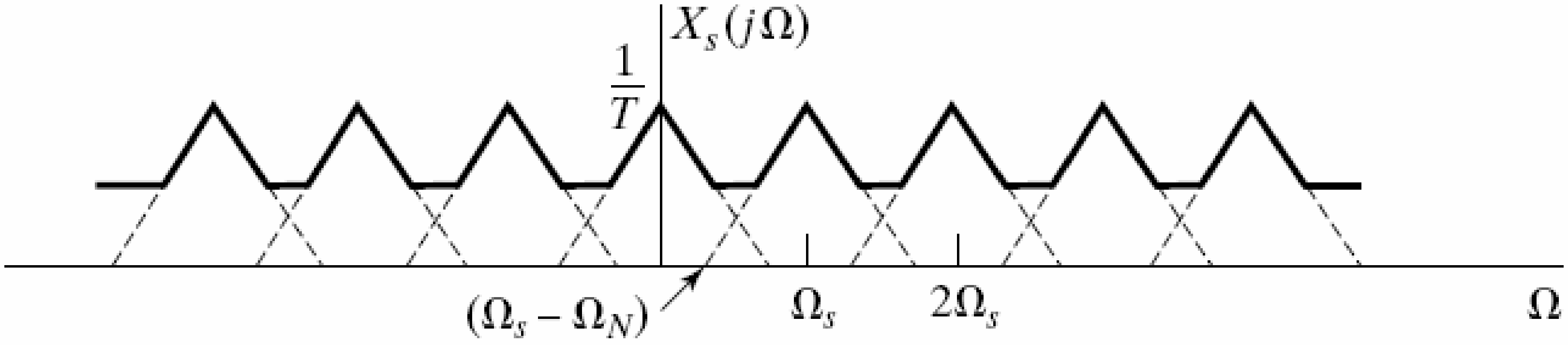
$S(j\Omega)$
$\frac{2\pi}{T}$
$-2\Omega_s$
$-\Omega_s$
0
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω

(b)



$X_s(j\Omega)$
$\frac{1}{T}$
$-2\Omega_s$
$-\Omega_s$
$-\Omega_N$
Ω_N
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω
$(\Omega_s - \Omega_N)$

(c)



(d)

Figure 4.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_S > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_S < 2\Omega_N$.

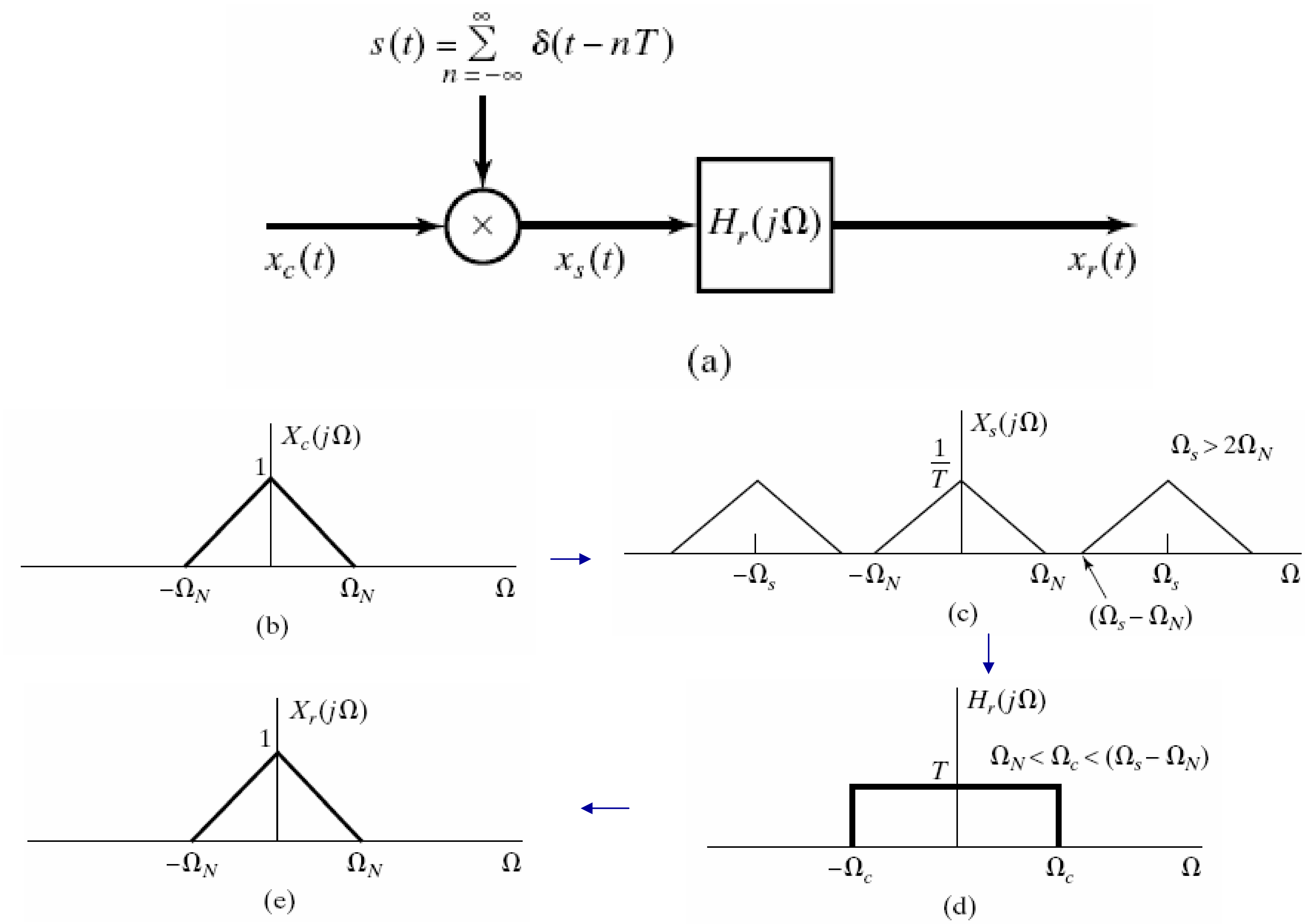


Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

Sampling of Bandpass Signals (c.f. Shenoi, 2006)

Suppose that we have an analog signal that is a bandpass signal (i.e., it has a Fourier transform that is zero outside the frequency range $\omega_1 \leq \omega \leq \omega_2$); the bandwidth of this signal is $B = \omega_2 - \omega_1$, and the maximum frequency of this signal is ω_2. So it is bandlimited, and according to Shannon's sampling theorem, one might consider a sampling frequency greater than $2\omega_2$; however, it is not necessary to choose a sampling frequency $\omega_s \geq 2\omega_2$ in order to ensure that we can reconstruct this signal from its sampled values. It has been shown [3] that when ω_2 is a multiple of B, we can recover the analog bandpass signal from its samples obtained with only a sampling frequency $\omega_s \geq 2B$. For example, when the bandpass signal has a Fourier transform between $\omega_1 = 4500$ and $\omega_2 = 5000$, we don't have to choose $\omega_s > 10,000$. We can choose $\omega_s > 1000$, since $\omega_2 = 10B$ in this example.

Correlation

- Given a pair of sequences $x[n]$ and $y[n]$, their cross correlation sequence is $r_{xy}[l]$ is defined as

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l] = x[n] * y[-l]$$

for all integer l. The cross correlation sequence can sometimes help to measure similarities between two signals.

It's very similar to convolution, unless the indices changes from $l - n$ to $n - l$.

- Autocorrelation:

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$

Properties

- Consider the following non-negative expression:

$$\sum_{n=-\infty}^{\infty}\left(ax[n]+y[n-l]\right)^2 = a^2\sum_{n=-\infty}^{\infty}x^2[n]+2a\sum_{n=-\infty}^{\infty}x[n]y[n-l]+\sum_{n=-\infty}^{\infty}y^2[n-l]$$

$$= a^2 r_{xx}[0]+2ar_{xy}[l]+r_{yy}[0]\geq 0$$

That is, $\begin{bmatrix} a & 1 \end{bmatrix}\begin{bmatrix} r_{xx}[0] & r_{xy}[l] \\ r_{xy}[l] & r_{yy}[0] \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix}\geq 0 \quad \text{for all } a$

- Thus, the matrix $\begin{bmatrix} r_{xx}[0] & r_{xy}[l] \\ r_{xy}[l] & r_{yy}[0] \end{bmatrix}$ is positive semidefinite.
  - Its determinate is nonnegative.

- The determinant is $r_{xx}[0]r_{yy}[0] - r_{xy}{}^2[l] \geq 0$.

Properties

$$r_{xx}[0]r_{yy}[0] \geq r_{xy}{}^2[l]$$
$$r_{xx}{}^2[0] \geq r_{xy}{}^2[l]$$

- Normalized cross correlation and autocorrelation:

$$\rho_{xx}[l] = \frac{r_{xx}[l]}{r_{xx}[0]} \qquad \rho_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0]r_{yy}[0]}}$$

The properties imply that $|\rho_{xx}[0]| \leq 1$ and $|\rho_{yy}[0]| \leq 1$.

- The DTFT of the autocorrelation signal $r_{xx}[l]$ is the squared magnitude of the DTFT of x[n], i.e., $|X(e^{jw})|^2$.

Correlation is useful in random signal processing

DFT and DTFT – A closer look

We discuss the DTFT-IDTFT pair ('I' means "inverse) for a discrete-time function given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

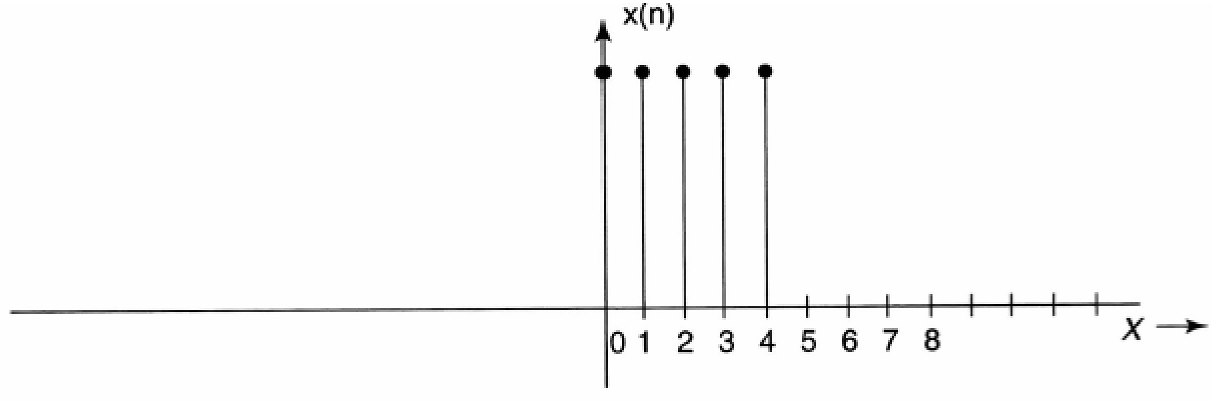
and

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

The pair and their properties and applications are elegant, but from practical point of view, we see some limitations; eg. the input signal is usually aperiodic and may be finite in length.

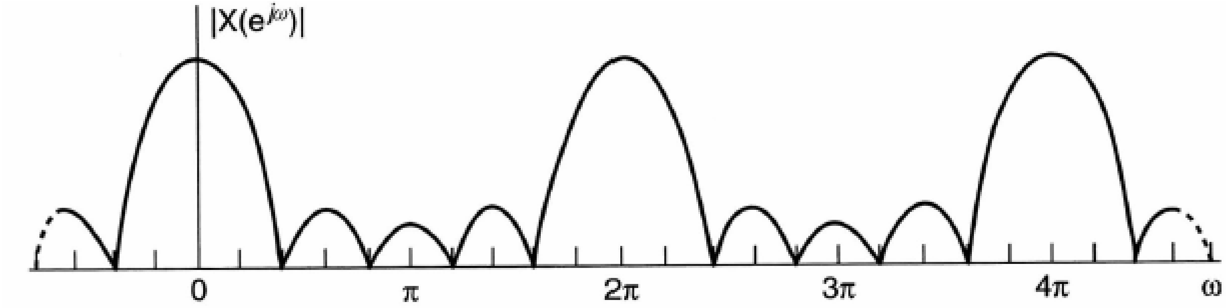
Example of a finite-length x(n) and its DTFT $X(e^{jw})$.

A finite-length signal



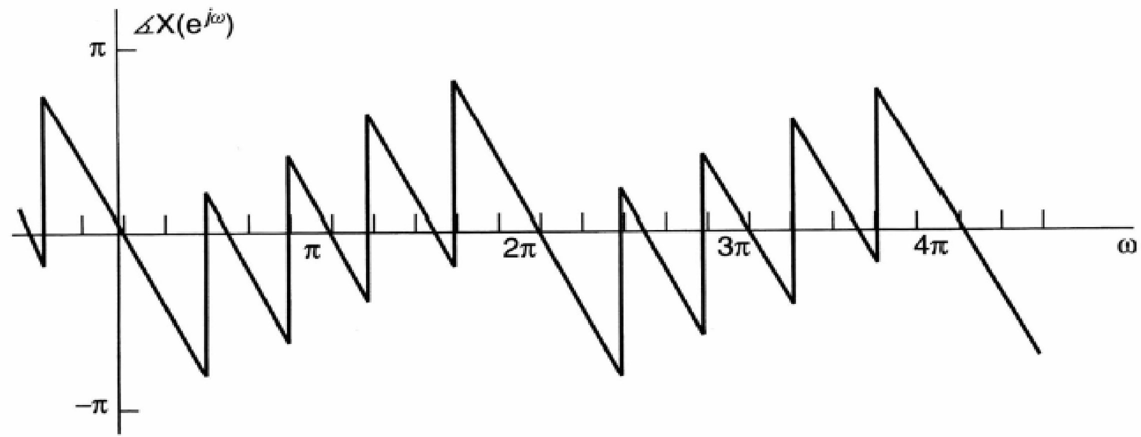
(a)

Its magnitude spectrum



(b)

Its phase spectrum



(c)

The function $X(e^{jw})$ is continuous in w, and the integration is not suitable for computation by a digital computer.

- We can discretize the frequency variable and find discrete values for $X(e^{jw_k})$, where w_k are equally sampled whthin [-π, π].

Discrete-time Fourier Series (DFS)

Let x(n) (n∈Z) be a finite-length sequence, with the length being N; i.e., x(n) = 0 for n < 0 and n > N.

Consider a periodic expansion of x(n):

$$x_p(n+KN) = x(n), \quad n = 0,1,\ldots,n-1, \quad K \text{ is any integer}$$

$x_p(n)$ is periodic, so it can be represented as a Fourier series:

$$x_p(n) = \sum_{k=0}^{N-1} X_p(k) e^{j(2\pi/N)kn}$$

To find the coefficients $X_p(k)$ (with respect to a discrete periodic signal), we use the following summation, instead of integration:

First, multiply both sides by e^{-jmw_0k}, and sum over n from n=0 to n=N-1:

$$\sum_{n=0}^{N-1} x_p(n) e^{-jm\omega_0 k} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X_p(k) e^{j(2\pi/N)kn} e^{-jm\omega_0 k}$$

By interchanging the order of summation, we get

$$\sum_{k=0}^{N-1} X_p(k) \left[\sum_{n=0}^{N-1} e^{j(2\pi/N)k(n-m)} \right]$$

Noting that

$[\sum_{n=0}^{N-1} e^{j(2\pi/N)k(n-m)}]$ is equal to N when $n = m$ and zero for all values of $n \neq m$.

pf. When n=m, the summation reduces to

$$[\sum_{n=0}^{N-1} e^{j0}] = N$$

When n≠m, by applying the geometric-sequence formula,

$$\sum_{n=-M}^{N} r^n = \frac{r^{N+1} - r^{-M}}{r-1}, \quad r \neq 1$$

we have

$$\sum_{n=0}^{N-1} e^{j(2\pi/N)k(n-m)} = \sum_{n'=-m}^{N-m-1} e^{j(2\pi/N)kn'} = \frac{e^{j(2\pi/N)k(N-m)} - e^{j(2\pi/N)k(-m)}}{e^{j(2\pi/N)k} - 1}$$

$$= \frac{e^{j(2\pi k)+j(2\pi/N)k(-m)} - e^{j(2\pi/N)k(-m)}}{e^{j(2\pi/N)k} - 1} = 0$$

Since there is only one nonzero term,

$$\sum_{k=0}^{N-1} X_p(k) \left[\sum_{n=0}^{N-1} e^{j(2\pi/N)k(n-m)} \right] = X_p(k)N$$

The final result is

$$X_p(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-jn\omega_0 k}$$

The following pairs then form the DFS

$$X_p(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-jn\omega_0 k}$$

$$x_p(n) = \sum_{k=0}^{N-1} X_p(k) e^{j(2\pi/N)kn}$$

Relation between DTFT and DFS for finite-length sequences

We note that

$$\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-jn\omega_0 k} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk}$$

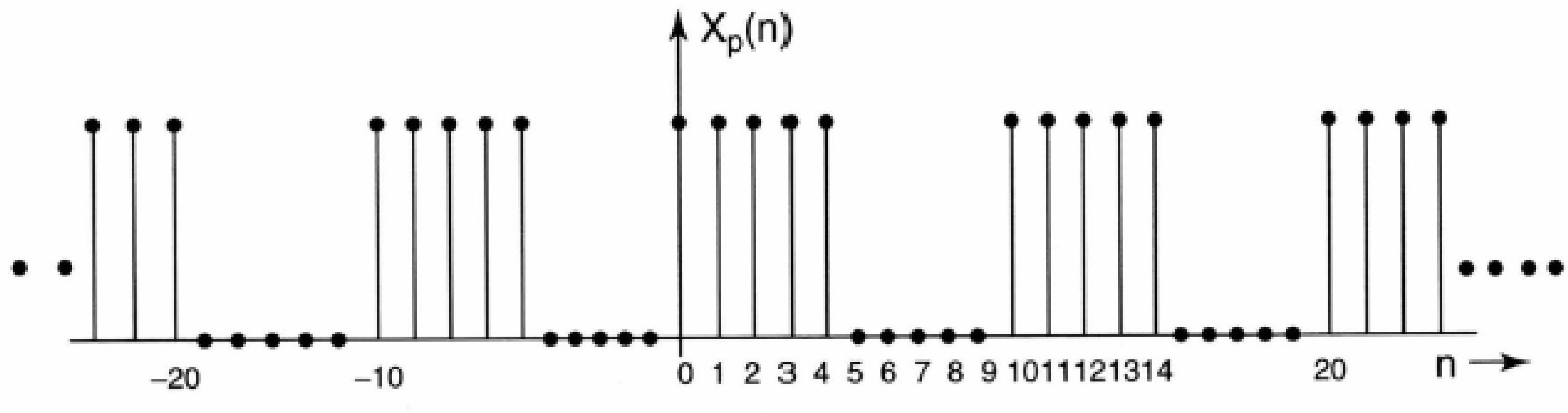
$$= \left(\frac{1}{N}\right) \underbrace{X_p(e^{j\omega})}_{\text{DTFT spectrum}}\Big|_{\omega_k=(2\pi/N)k} = \underbrace{X_p(k)}_{\text{DFS coeficient}}$$

DFS coeficient

DTFT spectrum

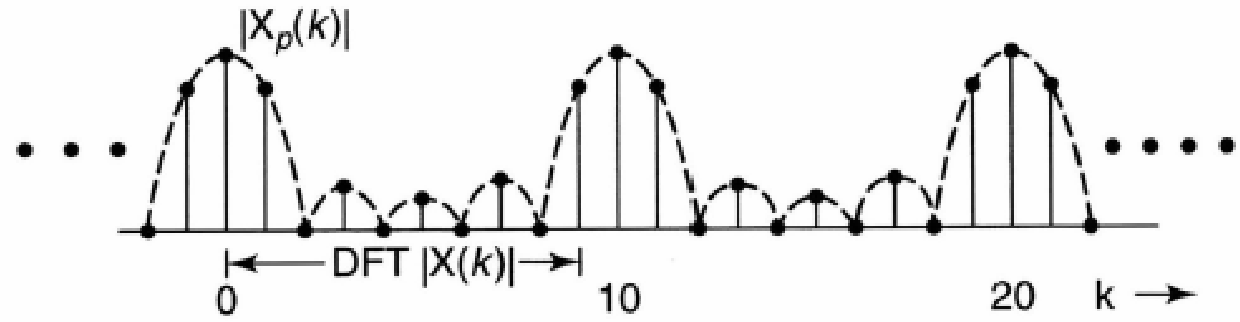
- In other words, when the DTFT of the finite length sequence x(n) is evaluated at the discrete frequency w_k = $(2\pi/N)k$, (which is the kth sample when the frequency range $[0, 2\pi]$ is divided into N equally spaced points) and dividing by N, we get the Fourier series coefficients $X_p(k)$.

A finite-length signal



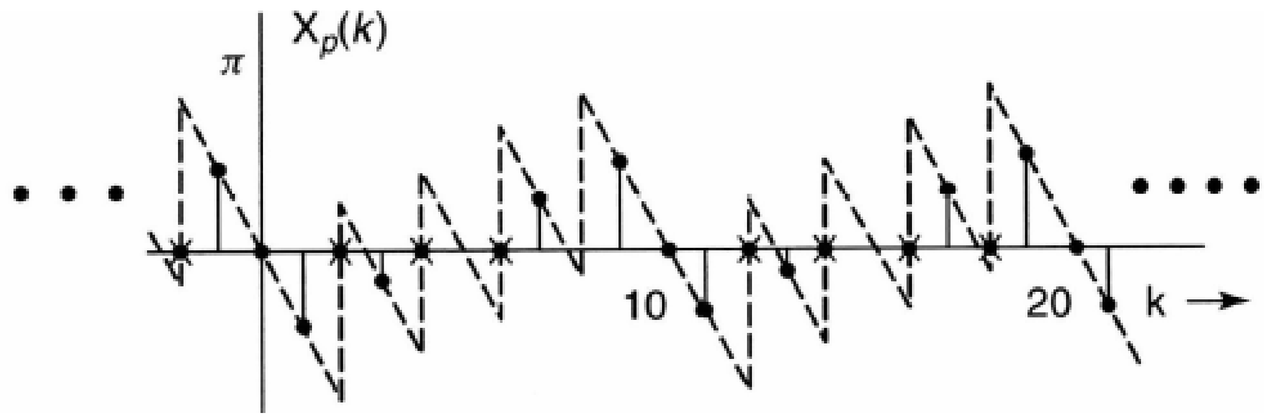
(a)

Its magnitu spectrum (sampled)



(b)

Its phase spectrum (sampled)



To simplify the notation, let us denote $W_N = e^{-j(2\pi/N)}$

The DFS-IDFS ('I' means "inverse") can be rewritten as ($W=W_N$)

$$x_p(n) = \sum_{k=0}^{N-1} X_p(k) W^{-kn}, \qquad -\infty < n < \infty$$

$$X_p(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) W^{kn}, \qquad -\infty < k < \infty$$

Discrete Fourier Transform (DFT)

- Consider both the signal and the spectrum only within one period (N-point signals both in time and frequency domains)

IDFT (inverse DFT)

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn} = \sum_{k=0}^{N-1} X(k) W^{-kn}, \qquad 0 \le n \le N-1$$

DFT

$$X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) W^{kn}, \qquad 0 \le k \le N-1$$

Relation between DFT and DTFT: The frequency coefficients of DFT is the N-point uniform samples of DTFT with $[0, 2\pi]$.

➢ The two equations DFT and IDFT give us a numerical algorithm to obtain the frequency response at least at the N discrete frequencies. By choosing a large value N, we get a fairly good idea of the frequency response for x(n), which is a function of the continuous variable w.

Question: Can we reconstruct the DTFT spectrum (continuous in w) from the DFT?

$\rightarrow$ Since the N-length signal can be exactly recovered from both the DFT coefficients and the DTFT spectrum, we expect that the DTFT spectrum (that is within $[0, 2\pi]$) can be exactly reconstructed by the DCT coefficients.

Reconstruct DTFT from DFT
(when the sequence is finite-length)

By substituting the inverse DFT into the x(n), we have

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi kn/N)}\right]e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k) \underbrace{\boxed{\sum_{n=0}^{N-1} e^{j(2\pi kn/N)}e^{-j\omega n}}}_{\text{a geometric sequence}}$$

a geometric sequence

By applying the geometric-sequence formula

$$\sum_{n=0}^{N-1} e^{j(2\pi kn/N)} e^{-j\omega n} = \frac{1 - e^{-j(\omega N - 2\pi k)}}{1 - e^{-j[\omega - (2\pi k/N)]}}$$

$$= \frac{e^{-j[(\omega N - 2\pi k)/2]}}{e^{-j[(\omega N - 2\pi k)/2N]}} \cdot \frac{\sin\left[\dfrac{\omega N - 2\pi k}{2}\right]}{\sin\left[\dfrac{\omega N - 2\pi k}{2N}\right]}$$

$$= \frac{\sin\left[\dfrac{\omega N - 2\pi k}{2}\right]}{\sin\left[\dfrac{\omega N - 2\pi k}{2N}\right]} e^{-j[\omega - (2\pi k/N)][(N-1)/2]}$$

So

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\frac{\sin\left[\dfrac{\omega N - 2\pi k}{2}\right]}{\sin\left[\dfrac{\omega N - 2\pi k}{2N}\right]} e^{-j[\omega-(2\pi k/N)][(N-1)/2]}$$

The reconstruction formula