DEFINING COMPLEX SINUSOIDS

Definitions

Complex sinusoids play a very important role in Electrical and Computer Engineering, especially communications and signal/image processing. The most general mathematical formula is:

$$\bar{x}(t) = Ae^{j(\omega_0 t+\phi)}$$

In this continuous-time signal, as in the real continuous-time sinusoid, the ***independent*** variable is time, t. The other variables indicate:

- The magnitude, $|\bar{x}(t)| = A$
- The angle, $\angle\bar{x}(t) = (\omega_0 t + \phi)$

As with the real sinusoid, $A > 0$ is called the amplitude, the frequency ω_0 is measured in "radians per second" and so is often called the "radian frequency," and the phase is ϕ. Using the Euler identity, we can write the complex sinusoid as

$$\bar{x}(t) = A\cos(\omega_0 t+\phi) + jA\sin(\omega_0 t+\phi)$$

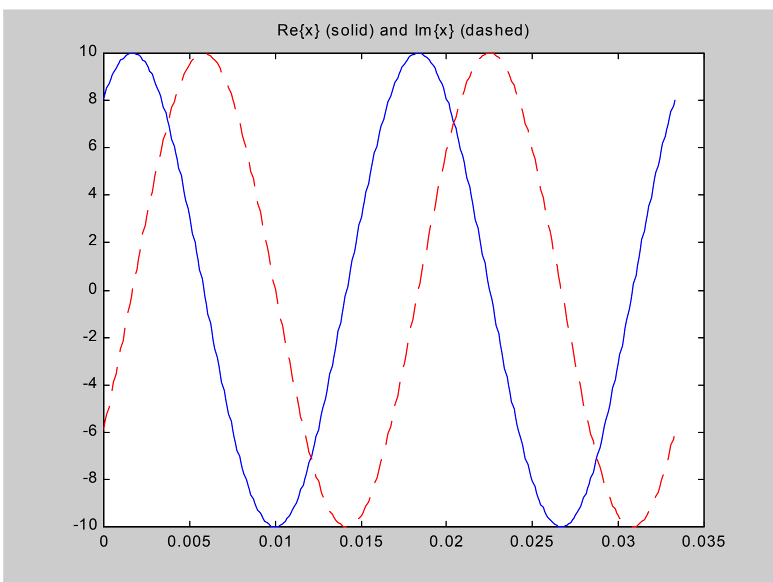
Consequently, we can see that the real cosine wave studied in the previous class could be written

$$x(t) = \mathrm{Re}\left\{Ae^{j(\omega_0 t+\phi)}\right\} = A\cos(\omega_0 t+\phi)$$

Consider the complex sinusoid

$$\bar{x}(t) = 10e^{j(2\pi 60t-0.2\pi)}$$

Since the signal is complex, we can plot the real part vs. time and the imaginary part vs. time:



The imaginary part of the signal is 90° ahead of the real part. Note from the description of the complex sinusoid that the frequency can be either "positive" or "negative." *What do the "positive" and "negative" frequencies represent physically?* Simply, they represent clockwise and counter-clockwise motion. More specifically for us, the complex representation will allow us to specify a specific quadrant in the polar plot, which means that we can exactly specify an angular value between 0 and 2π radians (or between $-\pi$ and π radians).

+ sin - cos	+ sin + cos
- sin - cos	- sin + cos

Furthermore, the complex exponential form will allow us to use simple algebra in place of remembering the cumbersome (and hard to remember) trigonometric identities that developed in the previous class meeting. For example, Consider that

$$\operatorname{Re}\left\{e^{j(\alpha+\beta)}\right\} = \cos(\alpha+\beta)$$

but also

$$= \operatorname{Re}\left\{e^{j\alpha}e^{j\beta}\right\}$$

$$= \operatorname{Re}\left\{(\cos\alpha + j\sin\alpha)(\cos\beta + j\sin\beta)\right\}$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

This will help us computationally. However, we have another reason also to consider the complex sinusoid notation, and that is the notion of the rotating "phasor."

Phasors

Define the "phasor transform" of the complex signal

$$\bar{x}(t) = Ae^{j(\omega_0 t+\phi)} \leftrightarrow X = Ae^{j\phi}$$

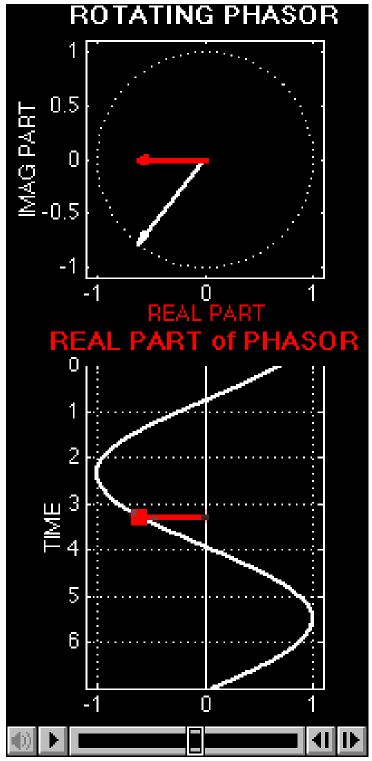
Notice that

$$\bar{x}(t) = Xe^{j\omega_0 t}$$

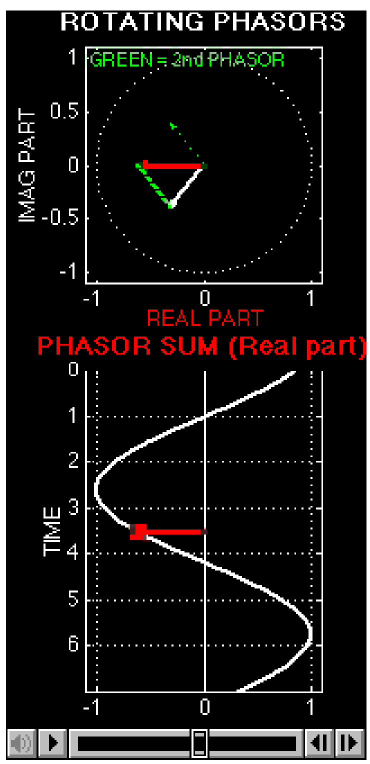
The complex value X is called the "complex amplitude" or phasor. One more useful definition is obtained from writing the complex sinusoid as

$$\bar{x}(t) = Ae^{j\theta(t)}$$

where the angle $\theta(t) \equiv w_0 t + \phi$. In the phasor transform, the frequency variable is not present (it has been "integrated out"). If we think of the phasor as a function of time, it rotates (either clockwise or counter-clockwise according to the sign of the frequency) once per period. Consider the rotating demo from the text:



Another way to visualize the Euler Identity is to examine the demonstration that is the sum of two complex conjugate rotating phasors:



In this example, consider the following:

$$\begin{aligned} A\cos(\omega_0 t+\phi) &= A\left(\frac{e^{j(\omega_0 t+\phi)}+e^{-j(\omega_0 t+\phi)}}{2}\right) \\ &= \frac{1}{2}Xe^{j\omega_0 t}+\frac{1}{2}X^* e^{-j\omega_0 t} \\ &= \frac{1}{2}\bar{x}(t)+\frac{1}{2}\bar{x}^*(t) \\ &= \mathrm{Re}\{\bar{x}(t)\} \end{aligned}$$

Thus, the real cosine of frequency ω_0 is actually the sum of two complex sinusoids with $\pm\omega_0$. These two complex sinusoids contribute half of the amplitude each.

Tuning Fork Example System

Newton's Law, $F = ma$, when applied to the tuning fork system described in the text becomes:

$$\begin{aligned} F &= ma \\ &= m\frac{d^2x(t)}{dt^2} \\ &= -kx(t) \end{aligned}$$

This yields the "differential equation"

$$\frac{d^2x(t)}{dt^2}+\frac{k}{m}x(t)=0$$

From our table of derivatives, we can guess that the solution is a sine wave. Suppose that we have $x(t)=\cos(\omega_0 t+\phi)$. Then

$$\frac{d^2x(t)}{dt^2}=-\omega_0^2\cos(\omega_0 t+\phi)$$

Substitution yields

$$\begin{aligned} -\omega_0^2\cos(\omega_0 t+\phi)+\frac{k}{m}\cos(\omega_0 t+\phi) &= 0 \\ -\omega_0^2+\frac{k}{m} &= 0 \\ \omega_0 &= \pm\sqrt{\frac{k}{m}} \end{aligned}$$

Consequently,

$$x(t)=\cos\left(\sqrt{\frac{k}{m}}t+\phi\right)$$

Now, consider the more general complex case where $x(t)=e^{j(\omega_0 t+\phi)}$. Taking the derivatives is straightforward

$$\frac{d^2x(t)}{dt^2}=-\omega_0^2 e^{j(\omega_0 t+\phi)}$$

Substitution yields

$$-\omega_0^2 e^{j(\omega_0 t+\phi)} + \frac{k}{m} e^{j(\omega_0 t+\phi)} = 0$$

$$\omega_0 = \pm\sqrt{\frac{k}{m}}$$

This example suggests that music (and other signals) can be created from sinusoids of varying frequencies (and amplitudes).