Defining real sinusoids

Sinusoids play a very important role in our discipline (Electrical and Computer Engineering). The most general mathematical formula is:

$$x(t) = A\cos(\omega_0 t + \phi)$$

In this continuous-time signal, the ***independent*** variable is time, t. The other variables indicate:

- Amplitude, A
- Frequency, ω_0
- Phase, ϕ

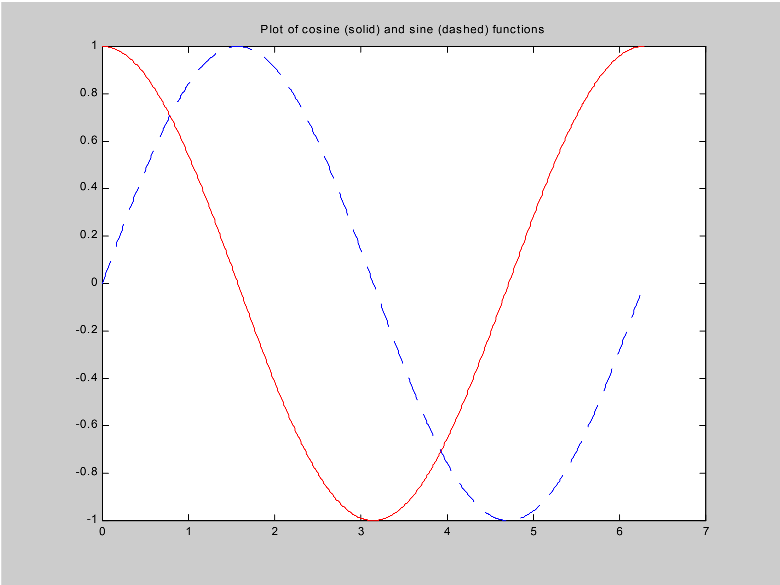
The frequency is measured in "radians per second" and so is often called the "radian frequency." To get the "frequency" f in Hz (defined to be sec^{-1}), we account for the 2π in

$$f_0 = \frac{\omega_0}{2\pi} \rightarrow \omega_0 = 2\pi f_0$$

Real sinusoids are "periodic" with period

$$T_0 = \frac{1}{f_0}$$

A plot of the cosine and sine functions over one period



reveals that the sine "lags" the cosine by 90 degrees, or that the cosine "leads" the sine by 90 degrees (ref: C. A. Gross, **Power Systems Analysis**, Wiley, p. 36). Mathematically, we say

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right) \text{ or } \cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

Other useful properties are:

- Periodicity: $\cos(\theta + 2\pi k) = \cos(\theta)$ for any integer k

- Cosine is even: $\cos(-\theta) = \cos(\theta)$
- Sine is odd: $\sin(-\theta) = -\sin(\theta)$
- Sine is zero: $\sin(\pi k) = 0$ for any integer k
- Sine has maximum amplitude: $\sin\left(\frac{\pi}{2}(2k+1)\right) = (-1)^k$ for any integer k
- Cosine is zero: $\cos\left(\frac{\pi}{2}(2k+1)\right) = 0$ for any integer k
- Cosine has maximum amplitude: $\cos(\pi k) = (-1)^k$ for any integer k

One reason that sinusoids are so important in Electrical and Computer Engineering is that the sine and cosine are derivatives of each other, namely

$$\frac{d\sin\theta}{d\theta} = \cos\theta \text{ and } \frac{d\cos\theta}{d\theta} = -\sin\theta$$

A few useful identities are

- $\sin^2\theta + \cos^2\theta = 1$
- $$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$
- $\sin(2\theta) = 2\cos\theta\sin\theta$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

As indicated before, the most general mathematical formula for a sinusoidal continuous-time signal is

$$\begin{aligned} x(t) &= A\cos(\omega_0 t + \phi) \\ &= A\cos(2\pi f_0 t + \phi) \end{aligned}$$

The time signal fluctuates between $\pm A$, the period of oscillation is $T = \frac{1}{f} = \frac{2\pi}{\omega}$, and the "initial" phase shift is given by ϕ. A periodic signal is one for which

$$x(t+T) = x(t)\ \forall t \text{ and for some } T$$

The least amount of time T for which the equality is true is called the "period." The period of oscillation for the sinusoid given above is easily proven using the definition above and the sum of angles identity (also given above)

$$\begin{aligned} x(t+T) &= A\cos(\omega_0(t+T) + \phi) \\ &= A\cos(\omega_0 t + \omega_0 T + \phi) \\ &= A\left[\cos(\omega_0 t + \phi)\cos(\omega_0 T) - \sin(\omega_0 t + \phi)\sin(\omega_0 T)\right] \end{aligned}$$

Now, if $T = \frac{2\pi}{\omega_0}$, $\cos(\omega_0 T) = \cos(2\pi) = 1$ and $\sin(\omega_0 T) = \sin(2\pi) = 0$. Substituting these values, we have

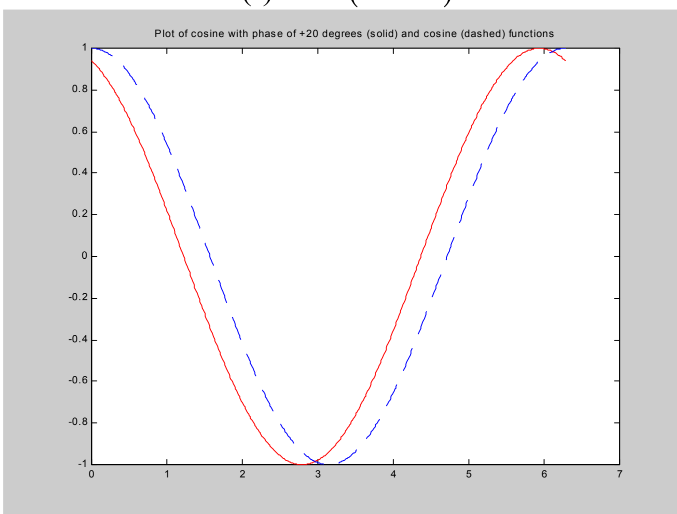
$$x(t+T) = A\cos(\omega_0 t + \phi) = x(t)$$

You should be able to verify that no smaller period T (other than the trivial answer of 0 seconds) will yield the necessary values of 1 and 0 needed to reduce the expanded sum of angles terms.

Time shift

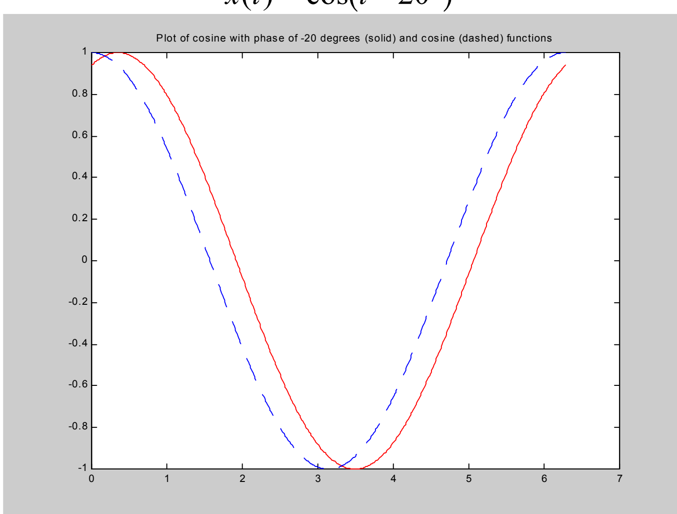
Now, it is necessary for us to consider (return to) the notion of time shift and its relationship to phase shift. Consider the signal

$$x(t) = \cos(t + 20°)$$



Note that the positive phase shift produces a time signal that is "ahead" of the regular (no phase shift included) cosine waveform. Thus, a positive phase shift results in an ***advanced*** time signal. Suppose that we change the sign on the phase shift

$$x(t) = \cos(t - 20°)$$



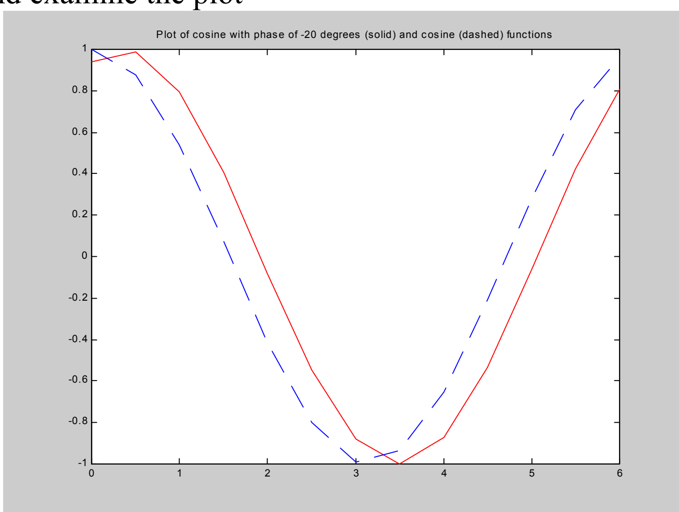
Here, the negative phase shift results in a ***delayed*** time signal. Now, notice that because the sines are periodic with period 2π, this could also be the graph of a time form that has been advanced 340 degrees. We could have delayed the time signal by 380 degrees. In fact, many potential advances or delays exist. We actually cannot tell just from these kinds of plots. The "principal value of the phase shift" is the angle between $\pm 180°$. Thus, all actual angles must be reduced ***modulo*** 360 degrees. This reduction is often called "phase wrapping," and sometimes the phase is "unwrapped" (see matlab and ECE 3793 for more details on this problem).

Plotting sinusoids in MATLAB

So far, we have only considered continuous-time sinusoids. We have plotted them in matlab, but I have assumed some knowledge needed to plot these continuous-time signals that must be explored. Let's examine the code used to produce the last sinusoid:

```
» n=0:0.01:2*pi;
» x=cos(n+2*pi/18);
» y=cos(n);
» plot(n,x,'r-',n,y,'b--')
» title('Plot of cosine with phase of +20 degrees (solid) and cosine
(dashed) functions')
```

Now, suppose that we change the "sampling time" in line 1 from 1/100th of a second to half of a second and examine the plot



Notice that the curves are no longer smooth. Matlab just connects the dots with straight lines, so a dense packing of lines is required to generate smooth plots. We will see in a later chapter what happens as we sample at points of time that are further and further apart.