USING COMPLEX NUMBERS

Three basic ideas:

- Operations on complex numbers obey the "normal" real rules with $j = \sqrt{-1}$.
- Trigonometry is eliminated in favor of algebra because of Euler's identity: $re^{j\theta} \overset{Euler}{=} r\cos\theta + jr\sin\theta$.
- Vectors in the complex phase plane represent complex numbers.

Notation and form conversions

Complex numbers may be written in rectangular form

$$\begin{aligned} z &= x + jy \\ &= \mathrm{Re}\{z\} + j\,\mathrm{Im}\{z\} \\ &= (x, y) \end{aligned}$$

polar form

$$z \leftrightarrow r\angle\theta$$

or as simple numbers (exponential form)

$$re^{j\theta}$$

Note that θ is usually given in degrees in the polar form, while it is a simple (radian) value in the exponential form. The exponential form is most commonly used because the normal rules of exponents apply. We have:

$$x = r\cos\theta, \text{ Real part}$$

$$y = r\sin\theta, \text{ Imaginary part}$$

$$r = \sqrt{x^2 + y^2}, \text{ Length (absolute value or magnitude) part}$$

$$\theta = \arctan\left(\frac{y}{x}\right), \text{ Direction (angle or phase) part}$$

Euler's Identity

The forward relationship:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

can easily be proven using McLaurin Series (see problem 2.4 which is a future homework problem). The inverse relations can be obtained by summing and subtracting:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Using this identity, a few important special cases are:

- $e^{j2\pi k} = \cos(2\pi k) + j\sin(2\pi k) = \cos(2\pi k) = 1$ for any integer k
- $e^{j\pi(2k+1)} = \cos\left(\pi(2k+1)\right) + j\sin\left(\pi(2k+1)\right) = \cos\left(\pi(2k+1)\right) = -1$ for any integer k
    - Thus, $e^{j\pi k} = (-1)^k$ for any integer k

- Similarly, $e^{j\frac{\pi}{2}(2k+1)} = (-1)^k j$

Operations

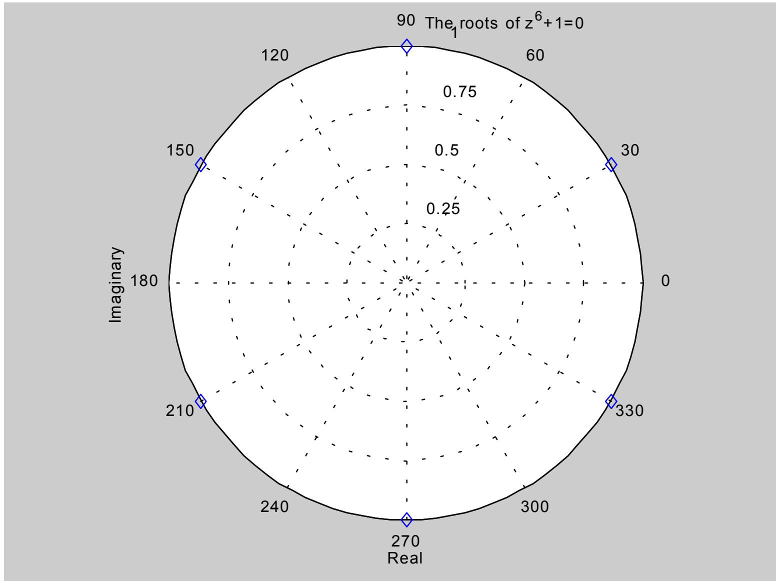
1. Addition (in rectangular form):
$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2) \end{aligned}$$
2. Subtraction (in rectangular form):
$$\begin{aligned} z_1 - z_2 &= (x_1 + jy_1) - (x_2 + jy_2) \\ &= (x_1 - x_2) + j(y_1 - y_2) \end{aligned}$$
3. Multiplication:
$$\begin{aligned} z_1 \times z_2 &= r_1 e^{j\theta_1} r_2 e^{j\theta_2} \\ &= r_1 r_2 e^{j(\theta_1+\theta_2)} \\ &= (x_1 + jy_1) \times (x_2 + jy_2) \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \end{aligned}$$
4. Conjugation:
$$\begin{aligned} z^* &= \left(re^{j\theta}\right)^* = re^{-j\theta} \\ &= x - jy \end{aligned}$$
5. Division:
$$\begin{aligned} z_1 \div z_2 &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} \\ &= \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2) + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \end{aligned}$$

Note that all forms are calculable from the rectangular form, though multiplication and division are not easy when compared to the exponential form. However, addition and subtraction must be performed in rectangular form, which necessitates the conversion of exponential forms according to the Euler identity.

Powers and roots

Consider the power
$$z^N = \left(re^{j\theta}\right)^N = r^N e^{j\theta N}$$
This suggests that
$$\sqrt[N]{z} = \sqrt[N]{re^{j\theta}} = r^{\frac{1}{N}} e^{j\theta\frac{1}{N}}$$
Then, the roots of the equation
$$z^N - c = 0$$
are found to be
$$|c|^{\frac{1}{N}} e^{j\frac{2\pi}{N}k},\ k = 0, 1, \ldots, N-1$$

For example, the roots of the equation $z^6 + 1 = 0$ are plotted



Then each root is 60° apart lying on the unit circle in the complex phase plane.