Shifting theorem for Z-transform

(1) for two side sequence $f(n) \longleftrightarrow F(Z)$ then

$f(n-m) \longleftrightarrow Z^{-m}F(Z), \qquad m$: positive integer

$f(n+m) \longleftrightarrow Z^{m}F(Z)$

pf:

$$Z^{-m}F(Z) = Z^{-m}\sum_{n=-\infty}^{\infty} f(n)Z^{-n} = \sum_{n=-\infty}^{\infty} f(n)Z^{-(m+n)}$$

$$= \sum_{n'=-\infty}^{\infty} f(n'-m)Z^{-n'}$$

(change variable $n+m=n'$). $f(n-m) \longleftrightarrow Z^{-m}F(Z)$

(2) for unilateral (one-side) sequence

$f(n)u(n) \longleftrightarrow F_1(Z)$ then

$f(n-n_0)u(n-n_0) \longleftrightarrow Z^{-n_0}F_1(Z)$ $\quad n_0$: positive integer

$f(n+n_0)u(n) \longleftrightarrow$

$Z^{n_0}[F_1(Z) - f(0) - f(1)Z^{-1} - ... - f(n_0-1)Z^{-(n_0-1)}]$

pf: (i)

$$Z^{-n_0}F_1(Z) = Z^{-n_0}\sum_{n=0}^{\infty} f(n)Z^{-n} = \sum_{n=0}^{\infty} f(n)Z^{-(n+n_0)}$$

$$= \sum_{n'=n_0}^{\infty} f(n'-n_0)Z^{-(n')} = \sum_{n'=0}^{\infty} f(n'-n_0)u(n'-n_0)Z^{-n'}$$

pf: (ii)

$$\sum_{n=0}^{\infty} f(n+n_0)Z^{-n} = \sum_{n'=n_0}^{\infty} f(n')Z^{-n'-n_0} = Z^{n_0} \sum_{n'=n_0}^{\infty} f(n')Z^{-n'}$$

$$= Z^{n_0}[\sum_{n'=0}^{\infty} f(n')Z^{-n'} - f(0) - f(1)Z^{-1} - \ldots - f(n_0-1)Z^{-(n_0-1)}]$$

Differentiation of Z-transform

$$f(n) \longleftrightarrow F(Z)$$
$$nf(n) \longleftrightarrow -Z\frac{dF(Z)}{dz}$$

pf:

$$\sum_{n=-\infty}^{\infty} nf(n)Z^{-n} = Z\sum_{n=-\infty}^{\infty} f(n)[nZ^{-(n+1)}]$$
$$= -Z\sum_{n=-\infty}^{\infty} f(n)[\frac{d}{dz}Z^{-n}]$$
$$= -Z\frac{d}{dz}[\sum_{n=-\infty}^{\infty} f(n)Z^{-n}] = -Z\frac{d}{dz}F(Z)$$

Convolution Theorem for Z-transform

(1) $x(n) \longleftrightarrow X(Z),\ y(n) \longleftrightarrow Y(Z)$
$x(n) * y(n) \longleftrightarrow X(Z)Y(Z)$

pf:
Let $w(n) = \sum_{n=-\infty}^{\infty} x(k)y(n-k)$, $w(n) \longleftrightarrow W(Z)$

$$W(Z) = \sum_{n=-\infty}^{\infty} w(n)Z^{-n} = \sum_{n=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} x(k)y(n-k)]Z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)Z^{-n}$$

(Let $n-k=m$: change variable)

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m) Z^{-(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x(k) Z^{-k} \sum_{m=-\infty}^{\infty} y(m) Z^{-m} = X(Z)Y(Z)$$

(2) $x(n)y(n) \longleftrightarrow \frac{1}{2\pi j}\oint X(\frac{Z}{\upsilon})Y(\upsilon)\upsilon^{-1}d\upsilon$

pf:

$w(n) = x(n)y(n)$,

$W(Z) = \sum_{n=-\infty}^{\infty} x(n)y(n)Z^{-n}y(n) = \frac{1}{2\pi j}\oint_{c_1} Y(\upsilon)\upsilon^{n-1}d\upsilon$

$$W(Z) = \sum_{n=-\infty}^{\infty} x(n)[\frac{1}{2\pi j}\oint_{c1} Y(\upsilon)\upsilon^{n-1}d\upsilon]Z^{-n}$$

$$= \frac{1}{2\pi j}\oint_{c_1}[\sum_{n=-\infty}^{\infty} x(n)(\frac{Z}{\upsilon})^{-n}]\upsilon^{-1}Y(\upsilon)d\upsilon$$

$$= \frac{1}{2\pi j}\oint_{c_1} X(\frac{Z}{\upsilon})Y(\upsilon)\upsilon^{-1}d\upsilon$$

Let $Z = e^{j\omega T} = e^{j\omega}$, set $T = 1$, and set $\upsilon = e^{j\theta}$
$X(Z)|_{Z=e^{j\omega}} = X(e^{j\omega})$; $Y(Z)|_{Z=e^{j\omega}} = Y(e^{j\omega})$
(Fourier transform)

then

$$W(e^{j\omega}) = \frac{1}{2\pi j}\int_{-\pi}^{\pi} Y(e^{j\theta})X(e^{j(\omega-\theta)})e^{-j\theta}je^{-j\theta}d\theta$$
$$= \frac{1}{2\pi j}\int_{-\pi}^{\pi} Y(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

convolution of two Fourier transforms

Initial value theorem for Z-transform

if $f(n)$ is a causal sequence, i.e., $f(n) = 0$ for $n < 0$ then

$$f(0) = \lim_{z\to\infty} F(Z)$$

pf:

$$\lim_{z\to\infty} F(Z) = \lim_{z\to\infty} [\sum_{n=-\infty}^{\infty} f(n)Z^{-n}]$$

$$= \lim_{z\to\infty} [\sum_{n=0}^{\infty} f(n)Z^{-n}] \qquad (f(n) = \text{causal})$$

$$= \lim_{z\to\infty} [f(0) + f(1)Z^{-1} + \cdots] = f(0)$$

Final value theorem

if $f(n)$ is a causal sequence i.e., $f(n) = 0$, $n < 0$
then

$$\lim_{n \to \infty} f(n) = \lim_{z \to 1} (Z - 1)F(z)$$

or

$$f(\infty) = \lim_{z \to 1} (Z - 1)F(Z)$$

pf:

$$z[f(n+1) - f(n)] = Z^{'}[f_1(Z) - f(0)] - F_1(Z) \text{ (shifting theorem)}$$
$$= (Z - 1)F_1(Z) - f(0)Z$$

$$z[f(n+1)-f(n)] = \sum_{n=0}^{\infty}[f(n+1)-f(n)]z^{-n}$$
$$= (Z-1)F_1 Z - f(0)Z$$

$$\lim_{z\to 1}\sum_{n=0}^{\infty}[f(n+1)-f(n)]Z^{-n} = \lim_{z\to 1}(Z-1)F_1(Z) - f(0)$$
$$= \lim_{n=0}^{\infty}[f(n+1)-f(n)]$$

$$\lim_{N\to\infty}\sum_{n=0}^{N}[f(n+1)-f(n)] = \lim_{z\to 1}(Z-1)F_1(Z)-f(0)$$

$$
\begin{array}{rl}
 & f(1)+f(2)+\cdots+f(n)+f(N+1) \\
+) & -f(0)-f(1)-f(2)-\cdots-f(N) \\
\hline
 & f(N+1)-f(0)
\end{array}
$$

$$\therefore \lim_{N\to\infty}\, [f(N+1) - \underbrace{f(0)}_{\approx 0}] = \lim_{z\to 1}\, (Z-1)F_(Z) - \underbrace{f(0)}_{\approx 0}$$

$$\Rightarrow \lim_{N\to\infty} f(N) = \lim_{Z\to 1}\, (Z-1)F_1(Z)$$

or $f(\infty) = \lim_{Z\to 1}\, (Z-1)F_1(Z)$

Z-transform Theorems

1. Shifting:
   if
   $$f(n) \leftrightarrow f(Z)$$
   then for any integer m
   $$f(n-m) \leftrightarrow Z^{-m}F(Z)$$

2. Convolution:
   if

$$f_1(n) \leftrightarrow F_1(Z)$$

   and

$$f_2(n) \leftrightarrow F_2(Z)$$

   , then

$$\sum_{k=-\infty}^{\infty} f_1(k) f_2(n-k) \leftrightarrow F_1(Z) F_2(Z)$$

Ex: Using Z-transform, we can find the sum of integers from 0 to n and the sum of their squares.

(a)

$$f_1(n) = \sum_{k=0}^{n} k = (nu(n)) * u(n) \leftrightarrow \frac{Z}{(Z-1)^2} \cdot \frac{Z}{(Z-1)}$$

$$= Z \cdot \frac{Z}{(Z-1)^3}$$

Since

$$\frac{Z}{(Z-1)^3} \leftrightarrow \frac{n(n-1)}{2} * u(n) \Rightarrow f_1(n) = \frac{n(n+1)}{2}$$

(b)

$$f_2(n) = \sum_{k=0}^{n} k^2 = (n^2 u(n)) * u(n) \leftrightarrow \frac{Z^3 + Z^2}{(Z-1)^4}$$

Since

$$n^2 u(n) = [2\frac{n(n-1)}{2} + n]u(n)$$

$$\overset{\text{linearity of Z-transform}}{\longleftrightarrow} \frac{2Z}{(Z-1)^3} + \frac{Z}{Z-1}$$

and

$$\binom{n}{3}u(n) \leftrightarrow \frac{Z}{(Z-1)^4}$$

$$f_2(n) = \frac{(n+2)(n+1)n}{6} + \frac{(n+1)n(n-1)}{6} = \frac{(2n+1)(n+1)n}{6}$$

3. Conjugate Sequences:
if

$$f(n) \leftrightarrow F(Z), \text{ then } f^*(-n) \leftrightarrow F^*(\frac{1}{Z^*})$$

Note: if the sequence $f(n)$ is real, then $F^*(\frac{1}{Z^*}) = F(\frac{1}{Z})$.
if $F(Z)$ converges in the ring, $r_1 < |Z| < r_2$, then

$F^*(\frac{1}{Z^*})$ converges in the ring $\frac{1}{r_2} < |Z| < \frac{1}{r_1}$.