Homework III
(計算過程務必清楚、明確，必要時可以加註說明)

1.  
a. The impulse response \(h[n]\) of a linear time-invariant system is known to be zero, except in the interval \(N_0 \leq n \leq N_1\). The input \(x[n]\) is known to be zero, except in the interval \(N_2 \leq n \leq N_3\). As a result, the output is constrained to be zero, except in some interval \(N_4 \leq n \leq N_5\). Determine \(N_4\) and \(N_5\) in terms of \(N_0, N_1, N_2\) and \(N_3\).

b. If \(x[n]\) is zero, except for \(N\) consecutive points, and \(h[n]\) is zero, except for \(M\) consecutive points, what is the maximum number of consecutive points for which \(y[n]\) can be nonzero?

2. Consider the linear constant-coefficient difference equation
\[y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1]\]
Determine \(y[n]\) for \(n \geq 0\) when \(x[n] = \delta[n]\) and \(y[n] = 0\), \(n < 0\)

3. A causal linear time-invariant system is described by the difference equation
\[y[n] - 5y[n-1] + 6y[n-2] = 2x[n-1]\]
a. Determine the **homogeneous response** of the system, i.e., the possible outputs if \(x[n] = 0\), \(\forall n\)
b. Determine the **impulse response** of the system. i.e., \(x[n] = \delta[n]\).
c. Determine the **step response** of the system. i.e., \(x[n] = u[n]\).

4.  
a. Find the **frequency response** \(H(e^{j\omega})\) of the linear time-invariant system whose input and output satisfy the difference equation
\[y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].\]
b. Write a **difference equation** that characterizes a system whose frequency response is
\[H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}\]

5. Determine the **output** of a linear time-invariant system if the impulse response \(h[n]\) and the input \(x[n]\) are as follows:
\[a. \quad x[n] = u[n] \text{ and } h[n] = a^n u[-n-1], \text{ with } a > 1\]
\[b. \quad x[n] = u[n-4] \text{ and } h[n] = 2^n u[-n-1]\]
\[c. \quad x[n] = u[n] \text{ and } h[n] = (0.5)2^n u[-n]\]
\[d. \quad h[n] = 2^n u[-n-1] \text{ and } x[n] = u[n] - u[n-10]\]
Use your knowledge of **linearity** and **time invariance** to minimize the work in Parts (b) – (d)
6. Consider a system with input \( x[n] \) and output \( y[n] \) that satisfy the difference equation

\[
y[n] = ny[n - 1] + x[n]
\]

The system is causal and satisfies initial-rest conditions; i.e., if \( x[n] = 0, \forall n < n_0 \), then \( y[n] = 0, \forall n < n_0 \).

a. If \( x[n] = \delta[n] \), determine \( y[n] \) for all \( n \)

b. Is the system linear? Justify your answer.

c. Is the system time invariant? Justify your answer.

7. Consider the system illustrated in the figure. The output of an LTI system with an impulse response \( h[n] = \left(\frac{1}{4}\right)^n u[n+10] \) is multiplied by a unit step function \( u[n] \) to yield the output of the overall system. Answer each of the following questions, and briefly justify your answers:

![System Diagram](image)

a. Is the overall system LTI?

b. Is the overall system causal?

c. Is the overall system stable in the BIBO sense?

8. For each of the following impulse responses of LTI systems, indicate whether or not the system is causal:

a. \( h[n] = \left(\frac{1}{2}\right)^n u[n] \)

b. \( h[n] = \left(\frac{1}{2}\right)^n u[n-1] \)

c. \( h[n] = \left(\frac{1}{2}\right)^{|n|} \)

d. \( h[n] = u[n+2] - u[n-2] \)

e. \( h[n] = \left(\frac{1}{3}\right)^n u[n] + 3^n u[-n-1] \)
9. For each of the following impulse responses of LTI systems, indicate whether or not the system is stable:
   a. \( h[n] = 4^n u[n] \)
   b. \( h[n] = u[n] - u[n-10] \)
   c. \( h[n] = 3^n u[-n-1] \)
   d. \( h[n] = \sin\left(\frac{n\pi}{3}\right) u[n] \)
   e. \( h[n] = \left(\frac{3}{4}\right)^n \cos\left(\frac{n\pi}{4} + \frac{\pi}{4}\right) \)
   f. \( h[n] = 2u[n+5]-u[n]-u[n-5] \)

10. Which of the following discrete-time signals could be eigenfunctions of any stable LTI system?
   a. \( 5^n u[n] \)
   b. \( e^{j2\pi n} \)
   c. \( e^{j\pi n} + e^{j2\pi n} \)
   d. \( 5^n \)
   e. \( 5^n \cdot e^{j2\pi n} \)

11. Determine which of the following signals is periodic. If a signal is periodic, determine its period.
   a. \( x[n] = e^{\left(\frac{2\pi n}{5}\right)} \)
   b. \( x[n] = \sin\left(\frac{m}{19}\right) \)
   c. \( x[n] = ne^{j\pi n} \)
   d. \( x[n] = e^{jn} \)

12. For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, and (4) time invariant.
   a. \( T(x[n]) = (\cos \omega n)x[n] \)
   b. \( T(x[n]) = x[n^2] \)
   c. \( T(x[n]) = x[n]\sum_{k=0}^{\infty} \delta[n-k] \)
   d. \( T(x[n]) = \sum_{k=n-1}^{\infty} x[k] \)

13. Consider a discrete-time linear time-invariant system with impulse response \( h[n] \). If the input \( x[n] \) is a periodic sequence with period \( N \) (i.e., if \( x[n] = x[n+N] \)), show that the output \( y[n] \) is also a periodic sequence with period \( N \).
14. Consider the linear time-invariant system with impulse response

\[ h[n] = \left( \frac{j}{2} \right)^n u[n], \quad \text{where } j = \sqrt{-1} \]

Determine the **steady-state response**, i.e., the response for large \( n \), to the excitation \( x[n] = \cos(\omega n) u[n] \)

15. Consider the three sequences

\[ v[n] = u[n] - u[n - 6] \]
\[ w[n] = \delta[n] + 2\delta[n - 2] + \delta[n - 4] \]
\[ q[n] = v[n] * w[n] \]

a. **Find and sketch** the sequence \( q[n] \)

b. **Find and sketch** the sequence \( r[n] \) such that \( r[n] * v[n] = \sum_{k=-\infty}^{n-1} q[k] \)

c. Is \( q[-n] = v[-n] * w[-n] \)? Justify your answer.

16. Consider a system \( S \) with input \( x[n] \) and output \( y[n] \) related according to the block diagram in the figure.

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\[ e^{-j\omega n} \]
\[ x[n] \rightarrow \times \rightarrow \text{LTI system} \quad h[n] \rightarrow y[n] \]
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The input \( x[n] \) is multiplied by \( e^{-j\omega n} \), and the product is passed through a **stable LTI** system with impulse response \( h[n] \).

a. Is the system \( S \) linear? Justify your answer.

b. Is the system \( S \) time invariant? Justify your answer.

c. Is the system \( S \) stable? Justify your answer.

d. Specify a system \( C \) such that the block diagram in the figure below represents an alternative way of expressing the input-output relationship of the system \( S \). *(Note: The system \( C \) does not have to be an LTI system.)*

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\[ x[n] \rightarrow h[n] e^{-j\omega n} \rightarrow C \rightarrow y[n] \]
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17. The **autocorrelation** sequence of a signal \( x[n] \) is defined as

\[ R_x[n] = \sum_{k=-\infty}^{\infty} x^*[k] x[n + k]. \]

a. Show that for an appropriate choice of the signal \( g[n] \), \( R_x[n] = x[n] * g[n] \), and **identify** the proper choice for \( g[n] \).

b. Show that the Fourier transform of \( R_x[n] \) is equal to \( |X(e^{j\omega})|^2 \).
18. Consider a discrete-time system with input $x[n]$ and output $y[n]$. When the input is

$$x[n] = \left(\frac{1}{4}\right)^n u[n],$$

the output is

$$y[n] = \left(\frac{1}{2}\right)^n, \forall n.$$

Determine which of the following statements is correct:

- ♦ The system must be LTI.
- ♦ The system could be LTI.
- ♦ The system cannot be LTI.

If your answer is that the system must or could be LTI, give a possible impulse response.

If your answer is that the system cannot be LTI, explain clearly why not.

19. A commonly used numerical operation called the **first backward difference** is defined as

$$y[n] = \nabla(x[n]) = x[n] - x[n-1],$$

where $x[n]$ is the input and $y[n]$ is the output of the first-backward-difference system.

a. Show that this system is linear and time invariant.

b. Find the impulse response of the system.

c. Find and sketch the frequency response (magnitude and phase).

d. Show that if

$$x[n] = f[n] * g[n]$$

then

$$\nabla(x[n]) = \nabla(f[n]) * g[n] = f[n] * \nabla(g[n])$$

Where * denotes discrete convolution.

e. Find the impulse response of a system that could be cascaded with the first-difference system to recover the input; i.e., find $h[n]$, where

$$h[n] * \nabla(x[n]) = x[n]$$

20. Let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$. Using the Fourier transform synthesis or analysis equations, show that

a. the Fourier transform of $x^*[n]$ is $X^*(e^{-j\omega})$,

b. the Fourier transform of $x^*[n-1]$ is $X^*(e^{-j\omega})$. 