Convergence Region of Z-transform

- The sum of the series may not be converge for all $z$.

**Region of convergence (ROC)**
- Since the $z$-transform can be interpreted as the Fourier transform, it is possible for the $z$-transform to converge even if the Fourier transform does not.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \leq \sum_{n=-\infty}^{\infty} |x[n]|z^{-n}$$
ROC of Z-transform

• In fact, convergence of the power series $X(z)$ depends only on $|z|$.

$$\sum_{n=-\infty}^{\infty} |x[n]| |z^{-n}| < \infty$$

• If some value of $z$, say $z = z_1$, is in the ROC, then all values of $z$ on the circle defined by $|z|=|z_1|$ will also be in the ROC.

• Thus the ROC consists of a ring in the z-plane.
ROC of Z-transform – Ring Shape
Analytic Function and ROC (advanced)

- The $z$-transform is a Laurent series of $z$.
  - A number of theorems from the complex-variable theory can be employed to study the $z$-transform.
  - A Laurent series, and therefore the $z$-transform, represents an *analytic* function at every point inside the region of convergence.
  - Hence, the $z$-transform and all its derivatives exist and must be continuous functions of $z$ with the ROC.
  - This implies that if the ROC includes the unit circle, the Fourier transform and all its derivatives with respect to $w$ must exist and is a continuous function of $w$. 
Example: Right-sided Exponential Sequence

- **Right-sided sequence:**
  - A discrete-time signal is right-sided if it is nonzero only for $n \geq 0$.
- Consider the signal $x[n] = a^n u[n]$.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- For convergent $X(z)$, we need $\sum_{n=0}^{\infty} (az^{-1})^n < \infty$

  - Thus, the ROC is the range of values of $z$ for which $|az^{-1}| < 1$ or, equivalently, $|z| > a$. 
Example: Right-sided Exponential Sequence (continue)

- By sum of power series,
  \[ X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a| \]
- There is one zero, at \( z=0 \), and one pole, at \( z=a \).

\[ X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a| \]

- \( \circ \) : zeros
- \( \times \) : poles

Gray region: ROC
Example: Left-sided Exponential Sequence

• Left-sided sequence:
  – A discrete-time signal is left-sided if it is nonzero only for \( n \leq -1 \).

• Consider the signal \( x[n] = -a^n u[-n-1] \).

\[
X(z) = - \sum_{n=-\infty}^{\infty} a^n u[-n-1]z^{-n} = - \sum_{n=-\infty}^{-1} a^n z^{-n} = \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n
\]

  – If \( |a z^{-1}| < 1 \) or, equivalently, \( |z| < a \), the sum converges.
Example: Left-sided Exponential Sequence (continue)

- By sum of power series,
  \[ X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{z}{z - a}, \quad |z| < |a| \]

- There is one zero, at \( z=0 \), and one pole, at \( z=a \).

The pole-zero plot and the algebraic expression of the system function are the same as those in the previous example, but the ROC is different.
Hence, to uniquely identify a sequence from its z-transform, we have to specify additionally the ROC of the z-transform.

Another example: given

\[ x(n) = \left( \frac{1}{2} \right)^n u(n) + \left( -\frac{1}{3} \right)^n u(n) \]

Then

\[ X(z) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2} \right)^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} \left( -\frac{1}{3} \right)^n u(n) z^{-n} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} + \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n z^{-n} \]

\[ = \frac{1}{1 - \frac{1}{2} \cdot z^{-1}} + \frac{1}{1 + \frac{1}{3} \cdot z^{-1}} = \frac{2z \left( z - \frac{1}{12} \right)}{\left( z - \frac{1}{2} \right) \left( z + \frac{1}{3} \right)} \]
\[
\left( \frac{1}{2} \right)^n u(n) \quad \leftrightarrow \quad z \quad \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| > \frac{1}{2}
\]

\[
\left( -\frac{1}{3} \right)^n u(n) \quad \leftrightarrow \quad z \quad \frac{1}{1 + \frac{1}{3} z^{-1}}, \quad |z| > \frac{1}{3}
\]

Thus

\[
\left( \frac{1}{2} \right)^n u(n) + \left( -\frac{1}{3} \right)^n u(n) \quad \leftrightarrow \quad z \quad \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}}, \quad |z| > \frac{1}{2}
\]
Consider another two-sided exponential sequence

Given

\[ x(n) = \left( -\frac{1}{3} \right)^n u(n) - \left( \frac{1}{2} \right)^n u(-n-1) \]

Since

\[ \left( -\frac{1}{3} \right)^n u(n) \leftrightarrow \frac{1}{1 + \frac{1}{3} z^{-1}}, \quad |z| > \frac{1}{3} \]

and by the left-sided sequence example

\[ -\left( \frac{1}{2} \right)^n u(-n-1) \leftrightarrow \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| < \frac{1}{2} \]
\[ X(z) = \frac{1}{1 + \frac{1}{3} z^{-1}} + \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{2z \left( z - \frac{1}{12} \right)}{(z + \frac{1}{3})(z - \frac{1}{2})} \]

Again, the poles and zeros are the same as the previous example, but the ROC is not.
Properties of the ROC

- The ROC is a ring or disk in the $z$-plane centered at the origin; i.e., $0 \leq r_R < |z| \leq r_L \leq \infty$.
- The Fourier transform of $x[n]$ converges absolutely iff the ROC includes the unit circle.
- The ROC cannot contain any poles.
- If $x[n]$ is a finite-length sequence (FIR), then the ROC is the entire $z$-plane except possible $z = 0$ or $z = \infty$.
- If $x[n]$ is a right-sided sequence, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly include) $z = \infty$. 
Properties of the ROC (continue)

• If $x[n]$ is a left-sided sequence, the ROC extends inward from the innermost (i.e., smallest magnitude) nonzero pole in $X(z)$ to (and possibly include) $z = 0$.

• A two-sided sequence $x[n]$ is an infinite-duration sequence that is neither right nor left sided. The ROC will consist of a ring in the $z$-plane, bounded on the interior and exterior by a pole, but not containing any poles.

• The ROC must be a connected region.
Example

A system with three poles
Different possibilities of the ROC. (b) ROC to a right-sided sequence. (c) ROC to a left-handed sequence.
Different possibilities of the ROC. (d) ROC to a two-sided sequence. (e) ROC to another two-sided sequence.
ROC vs. LTI System

• Consider the system function $H(z)$ of a linear system:
  – If the system is stable, the impulse response $h(n)$ is absolutely summable and therefore has a Fourier transform, then the ROC must include the unit circle.
  – If the system is causal, then the impulse response $h(n)$ is right-sided, and thus the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $H(z)$ to (and possibly include) $z=\infty$.
  – That is, a causal LTI system is stable if the poles are all inside the unit circle.
Inverse Z-transform

- Given $X(z)$, find the sequence $x[n]$ that has $X(z)$ as its z-transform.
- We need to specify both algebraic expression and ROC to make the inverse Z-transform unique.
- Techniques for finding the inverse z-transform:
  - Investigation method:
    - By inspect certain transform pairs.
    - Eg. If we need to find the inverse z-transform of
      \[ X(z) = \frac{1}{1 - 0.5z^{-1}} \]
      From the transform pair we see that $x[n] = 0.5^n u[n]$. 
Inverse Z-transform and Difference Equation

- Inverse Z-transform can help solve difference equations.
- Given a difference equation (an LTI system), we can represent its system function $H(Z)$ by Z-transform.
- Solving difference equation: for an input $x[n]$, we can find its output $y[n]$ by the inverse Z-transform of $Y(Z)$ (where $Y(Z) = X(Z)H(Z)$).
Some Common Z-transform Pairs

\[ \delta[n] \leftrightarrow 1 \quad \text{ROC: all } z. \]
\[ u[n] \leftrightarrow \frac{1}{1 - z^{-1}} \quad \text{ROC: } |z| > 1. \]
\[ -u[-n-1] \leftrightarrow \frac{1}{1 - z^{-1}} \quad \text{ROC: } |z| < 1. \]
\[ \delta[n-m] \leftrightarrow z^{-m} \quad \text{ROC: all } z \text{ except } 0 \text{ (if } m > 0 \text{) or } \infty \text{ (if } m < 0 \text{).} \]
\[ a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}} \quad \text{ROC: } |z| > |a|. \]
\[ -a^n u[-n-1] \leftrightarrow \frac{1}{1 - az^{-1}} \quad \text{ROC: } |z| < |a|. \]
\[ na^n u[n] \leftrightarrow \frac{az^{-1}}{(1 - az^{-1})^2} \quad \text{ROC: } |z| > |a|. \]
Some Common Z-transform Pairs (continue)

\[-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}\]

[\text{ROC} : |z| < |a|.

[\cos w_0 n u[n] \leftrightarrow \frac{1-[\cos w_0]z^{-1}}{1-[2\cos w_0]z^{-1} + z^{-2}}\]

[\text{ROC} : |z| > 1.

[\sin w_0 n u[n] \leftrightarrow \frac{[\sin w_0]z^{-1}}{1-[2\cos w_0]z^{-1} + z^{-2}}\]

[\text{ROC} : |z| > 1.

[\text{ROC} : |z| > r.]

[\left\{ \begin{array}{ll}
a^n & 0 \leq n \leq N - 1 \\
0 & \text{otherwise}
\end{array} \right. \leftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}}\]

[\text{ROC} : |z| > 0.]
Inverse $Z$-transform by Partial Fraction Expansion

- If $X(z)$ is the rational form with

\[
X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}
\]

- An equivalent expression is

\[
X(z) = \frac{z^{-M} \sum_{m=0}^{M} b_m z^{M-m}}{z^{-N} \sum_{k=0}^{N} a_k z^{N-k}} = \frac{z^{M} \sum_{m=0}^{M} b_m z^{M-m}}{z^{N} \sum_{k=0}^{N} a_k z^{N-k}}
\]
Inverse Z-transform by Partial Fraction Expansion (continue)

- There will be $M$ zeros and $N$ poles at nonzero locations in the $z$-plane.
- Note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \prod_{m=1}^{M} \frac{1 - c_m z^{-1}}{\prod_{m=1}^{N} (1 - d_k z^{-1})}$$

where $c_k$'s and $d_k$'s are the nonzero zeros and poles, respectively.
Inverse Z-transform by Partial Fraction Expansion (continue)

Then \( X(z) \) can be expressed as

\[
X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}
\]

Obviously, the common denominators of the fractions in the above two equations are the same. Multiplying both sides of the above equation by \( 1 - d_k z^{-1} \) and evaluating for \( z = d_k \) shows that

\[
A_k = \left(1 - d_k z^{-1}\right)X(z) \bigg|_{z=d_k}
\]

Example

- Find the inverse z-transform of

\[ X(z) = \left( \frac{1}{1 - \left(\frac{1}{4}\right)z^{-1}} \right) \left( \frac{1}{1 - \left(\frac{1}{2}\right)z^{-1}} \right) \quad |z| > \frac{1}{2} \]

\( X(z) \) can be decomposed as

\[ X(z) = \left( \frac{A_1}{1 - \left(\frac{1}{4}\right)z^{-1}} \right) + \left( \frac{A_2}{1 - \left(\frac{1}{2}\right)z^{-1}} \right) \]

Then

\[ A_1 = \left(1 - \left(\frac{1}{4}\right)z^{-1}\right)X(z) \bigg|_{z = \frac{1}{4}} = -1 \]

\[ A_2 = \left(1 - \left(\frac{1}{2}\right)z^{-1}\right)X(z) \bigg|_{z = \frac{1}{2}} = 2 \]
Example (continue)

• Thus

\[ X(z) = \frac{-1}{1 - (1/4)z^{-1}} + \frac{2}{1 - (1/2)z^{-1}} \]

From the ROC if we have a right-hand sequence,

\[ x[n] = 2 \left( \frac{1}{2} \right)^n u[n] - \left( \frac{1}{4} \right)^n u[n] \]
Another Example

• Find the inverse z-transform of

\[ X(z) = \frac{(1+z^{-1})^2}{(1-(1/2)z^{-1})(1-z^{-1})} \quad |z| > 1 \]

Since both the numerator and denominator are of degree 2, a constant term exists.

\[ X(z) = B_0 + \frac{A_1}{1-(1/2)z^{-1}} + \frac{A_2}{1-z^{-1}} \]

\(B_0\) can be determined by the fraction of the coefficients of \(z^{-2}\), \(B_0 = 1/(1/2) = 2.\)
Another Example (continue)

\[ X(z) = 2 + \frac{A_1}{(1-(1/2)z^{-1})} + \frac{A_2}{(1-z^{-1})} \]

\[ A_1 = 2 + \frac{-1+5z^{-1}}{(1-(1/2)z^{-1})(1-z^{-1})} \left(1-(1/2)z^{-1}\right) \bigg|_{z=1/2} = 9 \]

\[ A_2 = 2 + \frac{-1+5z^{-1}}{(1-(1/2)z^{-1})(1-z^{-1})} \left(1-z^{-1}\right) \bigg|_{z=1} = 8 \]

From the ROC, the solution is right-handed. So

\[ X(z) = 2 - \frac{9}{(1-(1/2)z^{-1})} + \frac{8}{(1-z^{-1})} \]

\[ x[n] = 2\delta[n] - 9(1/2)^n u[n] + 8u[n] \]
We can determine any particular value of the sequence by finding the coefficient of the appropriate power of $z^{-1}$.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

\[
= ... + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + ...
\]
Example: Finite-length Sequence

- Find the inverse z-transform of

\[ X(z) = z^2 \left(1 - 0.5z^{-1}\right) \left(1 + z^{-1}\right) \left(1 - z^{-1}\right) \]

By directly expand \( X(z) \), we have

\[ X(z) = z^2 - 0.5z - 1 + 0.5z^{-1} \]

Thus,

\[ x[n] = \delta[n+2] - 0.5\delta[n+1] - \delta[n] + 0.5\delta[n-1] \]
Example (advanced)

- Find the inverse z-transform of

\[ X(z) = \log(1 + az^{-1}) \quad |z| > |a| \]

Using the power series expansion for \( \log(1+x) \) with \( |x| < 1 \), we obtain

\[ X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \]

Thus

\[ x[n] = \begin{cases} 
(-1)^{n+1} \frac{a^n}{n} & n \geq 1 \\
0 & n \leq 0 
\end{cases} \]
Z-transform Properties

• Suppose

\[ x[n] \overset{z}{\leftrightarrow} X(z) \quad \text{ROC} = R_x \]

\[ x_1[n] \overset{z}{\leftrightarrow} X_1(z) \quad \text{ROC} = R_{x_1} \]

\[ x_2[n] \overset{z}{\leftrightarrow} X_2(z) \quad \text{ROC} = R_{x_2} \]

• Linearity

\[ ax_1[n] + b \ x_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z) \quad \text{ROC} = R_{x_1} \cap R_{x_2} \]
Z-transform Properties (continue)

- **Time shifting**

\[ x[n - n_0] \overset{z}{\leftrightarrow} z^{-n_0} X(z) \quad \text{ROC} = R_x \]  
(except for the possible addition or deletion of \( z=0 \) or \( z=\infty \).)

- **Multiplication by an exponential sequence**

\[ z_0^n x[n] \overset{z}{\leftrightarrow} X(z / z_0) \quad \text{ROC} = \left| z_0 \right| R_x \]
Z-transform Properties (continue)

- Differentiation of $X(z)$

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \quad \text{ROC} = R_x$$

- Conjugation of a complex sequence

$$x^*[n] \leftrightarrow X^*(z^*) \quad \text{ROC} = R_x$$
Z-transform Properties (continue)

- Time reversal

\[ x[n] \overset{z}{\longleftrightarrow} X(z) \]

If the sequence is real, the result becomes

\[ x[-n] \overset{z}{\longleftrightarrow} X(1/z) \quad \text{ROC} = \frac{1}{R_x} \]

- Convolution

\[ x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z) \quad \text{ROC contains } R_{x_1} \cap R_{x_2} \]
Initial-value theorem: If \( x[n] \) is zero for \( n < 0 \) (i.e., if \( x[n] \) is causal), then

\[
x[0] = \lim_{z \to \infty} X(z)
\]
Z-transform Properties (advanced)

- General formula of inverse z-transform:

\[
\mathcal{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}
\]

\[
Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint X(z)z^{n-1} \, dz
\]
Manipulate Frequencies in Discrete-time Domain

- **Three common ways:**
  - FFT/IFFT
  - Difference equation/Z-transform
  - Down-sampling/up-sampling (will be introduced later)
Changing the Sampling rate using discrete-time processing

- What happens when sampling in the discrete domain?

\[ x_d[n] = x[nM] \]

- **downsampling**; sampling rate compressor;

\[ x_d[n] = x[nM] \]
Frequency domain of downsampling

- This is a ‘re-sampling’ process. Remember from continuous-time sampling of \( x[n] = x_c(nT) \), we have

\[
X(e^{jw}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j\left( \frac{w}{T} - \frac{2\pi k}{T} \right) \right)
\]

- Hence, for the down-sampled signal \( x_d[m] = x_c(mT') \), (where \( T' = MT \)), we have

\[
X_d(e^{jw}) = \frac{1}{T'} \sum_{r=-\infty}^{\infty} X_c \left( j\left( \frac{w}{T'} - \frac{2\pi r}{T'} \right) \right)
\]
Frequency domain of downsampling

- We are interested in the relation between $X(e^{jw})$ and $X_d(e^{jw})$.
- Let's represent $r$ as $r = i + kM$, where $0 \leq i \leq M-1$, (i.e., $r \equiv i \pmod{M}$). Then

$$X_d(e^{jw}) = \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c \left( j \left( \frac{w}{MT} - \frac{2\pi r}{MT} \right) \right)$$

$$= \frac{1}{M} \sum_{i=0}^{M-1} \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j \left( \frac{w}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT} \right) \right)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j \left( \frac{w-2\pi i}{MT} - \frac{2\pi k}{T} \right) \right) = X(e^{j(w-2\pi i)/M})$$
Frequency domain of downsampling

Therefore, the downsampling can be treated as a re-sampling process. Its frequency domain relationship is similar to that of the D/C converter as:

\[ X_d(e^{jw}) = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(w-2\pi i)/M}) \]

This is equivalent to compositing M copies of the of \( X(e^{jw}) \), frequency scaled by M and shifted by integer multiples of \( 2\pi/M \).

Note that downsampling is a linear system, but not an LTI system.

So, it does not have the “frequency response,” but we can still see its influence in the frequency domain as shown above.
Downsampling in the Frequency domain (without aliasing)

Downsampling the discrete-time signal by 2 (M=2)
(Assume $W_N = \pi/2$)

In the DTFT domain, when downsampling the discrete-time signal by 2, the frequency domain representation changes as shown in the diagram. The DTFT of the original signal $X(e^{j\omega})$ is modified to $X_d(e^{j\omega})$ according to the downsampling formula:

$$X_d(e^{j\omega}) = \frac{1}{MT} \left[ X(e^{j\omega/2}) + X(e^{j(\omega-2\pi)/2}) \right]$$
Downsampling in the Frequency domain (with aliasing)

Downsampling the discrete-time signal by 3 (M=3)
(Assume $W_N = \pi/2$)
Downsampling with prefiltering to avoid aliasing (decimation)

- From the above, the DTFT of the down-sampled signal is the superposition of \( M \) shifted/scaled versions of the DTFT of the original signal.
- To avoid aliasing, we need \( w_N < \pi/M \), where \( w_N \) is the highest frequency of the discrete-time signal \( x[n] \).
- Hence, downsampling is usually accompanied with a pre-low-pass filtering, and a low-pass filter followed by down-sampling is usually called a decimator, and termed the process as decimation.
(d) \[ H_d(e^{j\omega}) \]\n
\[ \omega_c = \frac{\pi}{M} \]

\[ \Omega T \]

(e) \[ \tilde{X}(e^{j\omega}) = H_d(e^{j\omega})X(e^{j\omega}) \]

\[ \frac{1}{T} \]

\[ \omega = \frac{\pi}{3} \]

\[ \frac{\pi}{M} = \frac{\pi}{3} \]

\[ \pi \]

\[ 2\pi \]

\[ \Omega T \]

(M = 3)
Up-sampling (or Expansion)

- Upsampling; sampling rate expander.

\[ x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases} \]

or equivalently,

\[ x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \]

Also, upsampling is a linear system, but not an LTI system.
Up-sampling example

- \[ x = [1 \ 2 \ 3 \ 4]; \]
- \[ y = \text{upsample}(x,3); \]
- Display x,y
  - x = 1 2 3 4
  - y = 1 0 0 2 0 0 3 0 0 4 0 0
Up-sampling (frequency domain)

- In frequency domain:

\[ X_e(e^{jw}) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (x[k] \delta[n - kL]) e^{-jwn} \]

\[ = \sum_{k=-\infty}^{\infty} (x[k] e^{-jwLk}) \]

\[ = X(e^{jwL}) \]
Example of up-sampling

Applying the sample frequency $2\pi/T \geq 2\Omega_N$, choose $T = \pi/\Omega_N$

Upsampling in the frequency domain
Up-sampling with post low-pass filtering

- There is no information loss in up-sampling. Thus the original signal can be reconstructed (by filtering).
- Upsampling is usually accompanied with a low-pass filter with cutoff frequency $\pi/L$ and gain $L$, to reconstruct the sequence.
- A low-pass filter followed by up-sampling is called an interpolator, and the whole process is called interpolation.
Example of up-sampling followed by low-pass filtering.

Applying low-pass filtering
Interpolation (time domain)

- If we choose an ideal lowpass filter with cutoff frequency $\pi/L$ and gain $L$, its impulse response is

$$h_i[n] = \frac{\sin(\pi n / L)}{\pi n / L}$$

- Hence

$$x_i[n] = x_e[n] * h_i[n] = \left( \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \right) * h_i[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n - kL) / L]}{\pi(n - kL) / L}$$

Its an interpolation of the discrete sequence $x[k]$
Sampling rate conversion by a non-integer rational factor

- By combining the decimation and interpolation, we can change the sampling rate of a sequence.
  - Changing the sampling rate by a non-integer factor $T' = TM/L$.
  - Eg., $L=100$ and $M=101$, then $T' = 1.01T$. 
Since the interpolation and decimation filters are in cascade, they can be combined as shown above.
Pre-filtering to avoid aliasing

- It is generally desirable to minimize the sampling rate.
- Eg., in processing speech signals, where often only the low-frequency band up to about 3-4k Hz is required, even though the speech signal may have significant frequency content in the 4k to 20k Hz range.
- Also, even if the signal is naturally bandlimited, wideband additive noise may fall in the higher frequency range, and as a result of sampling. These noise components would be aliased into the low frequency band.
Over-sampled A/D conversion

- The anti-aliasing filter is an analog filter. However, in applications involving powerful, but inexpensive, digital processors, these continuous-time filters may account for a major part of the cost of a system.

- Let $\Omega_N$ be the highest frequency of the analog signal. Instead, we first apply a very simple anti-aliasing filter that has a gradual cutoff (instead of a sharp cutoff) with significant attenuation at $M\Omega_N$. Next, implement the continuous-to-discrete (C/D) conversion at the sampling rate higher than $2M\Omega_N$.

- After that, sampling rate reduction by a factor of $M$ that includes sharp anti-aliasing filtering is implemented in the discrete-time domain.
Using over-sampled A/D conversion to simplify a continuous-time anti-aliasing filter
Example of over-sampled A/D conversion (analog domain)
Example of over-sampled A/D conversion (discrete-time domain)
Oversampling vs. quantization
(Oppenheim, Chap. 4)

- We consider the analog signal \( x_a(t) \) as zero-mean, wide-sense-stationary, random process with power-spectral density denoted by \( \Phi_{x_ax_a}(e^{jw}) \) and the autocorrelation function by \( \phi_{x_ax_a}(\tau) \).

- To simplify our discussion, assume that \( x_a(t) \) is already bandlimited to \( \Omega_N \), i.e.,

\[
\Phi_{x_ax_a}(j\Omega) = 0, \quad |\Omega| \geq \Omega_N,
\]
Oversampling: We assume that $\frac{2\pi}{T} = 2M\Omega_N$.

- $M$ is an integer, called the **oversampling ratio**.
Using the additive noise model, the system can be replaced by

Its output \( x_d[n] \) has two components, one from the signal input \( x_a(t) \) and the other from the quantization noise input \( e[n] \). Denote them as \( x_{da}[n] \) and \( x_{de}[n] \), respectively.
Signal component (assume e[n]=0)

- Goal: determine the signal-to noise ratio of signal power $\varepsilon\{x_{da}^2\}$ to the quantization-noise power $\varepsilon\{x_{de}^2\}$. ($\varepsilon\{.\}$ denotes the expectation value.)
- As $x_a(t)$ is converted into $x[n]$, and then $x_{da}[n]$, we focus on the power of $x[n]$ first.
- Let us analyze this in the time domain. Denote $\phi_{xx}[n]$ and $\Phi_{xx}(e^{jw})$ to be the autocorrelation and power spectral density of $x[n]$, respectively.
- By definition, $\phi_{xx}[m] = \varepsilon\{x[n+m]x[n]\}$. 
Power of $x[n]$ (assume $e[n]=0$)

- Since $x[n] = x_a(nT)$, it is easy to see that

$$
\phi_{xx}[m] = \varepsilon\{x[n + m]x[n]\} \\
= \varepsilon\{x_a((n + m)T)x_a(nT)\} \\
= \phi_{x_a x_a}(mT)
$$

- That is, the autocorrelation function of the sequence of samples is a sampled version of the autocorrelation function.

- The wide-sense-stationary assumption implies that $\varepsilon\{x_a^2(t)\}$ is a constant independent of $t$. It then follows that

$$
\varepsilon\{x^2[n]\} = \varepsilon\{x_a^2(nT)\} = \varepsilon\{x_a^2(t)\}
$$

for all $n$ or $t$. 
Power of $x_{da}[n]$ (assume $e[n]=0$)

- Since the decimation filter is an ideal lowpass filter with cutoff frequency $w_c = \pi/M$, the signal $x[n]$ passes unaltered through the filter.
- Therefore, the downsampled signal component at the output, $x_{da}[n]=x[nM]=x_a(nMT)$, also has the same power.
- In sum, the above analyses show that

$$\mathcal{E}\{x_{da}^2[n]\} = \mathcal{E}\{x^2[n]\} = \mathcal{E}\{x_a^2(t)\}$$

which shows that the power of the signal component stays the same as it traverses the entire system from the input $x_a(t)$ to the corresponding output component $x_{da}[n]$. 
According to previous studies, let us assume that $e[n]$ is a wide-sense-stationary white-noise process with zero mean and variance

$$\sigma_e^2 = \frac{\Delta^2}{12}$$

Consequently, the autocorrelation function and power density spectrum for $e[n]$ are,

$$\phi_{ee}[n] = \sigma_e^2 \delta[n]$$

The power spectral density is the DTFT of the autocorrelation function. So,

$$\Phi_{ee}(e^{jw}) = \sigma_e^2, \quad -\pi < w < \pi$$
Although we have shown that the power in $x_{da}[n]$ does not depend on $M$, we will show that the noise component $x_{de}[n]$ does not keep the same noise power.

It is because that, as the oversampling ratio $M$ increases, less of the quantization noise spectrum overlaps with the signal spectrum, as shown below.
Review of Downsampling in the Frequency domain (without aliasing)

Down-sampling

(power remains the same for the integral from \(-\pi\) to \(\pi\).)

(over) sampling

\[ X_d(e^{j\omega}) = \frac{1}{2} \left[ X(e^{j\omega/2}) + X(e^{j(\omega - 2\pi)/2}) \right] \]
So, when oversampled by \( M \), the power spectrum of \( x_a(t) \) and \( x[n] \) in the frequency domain are illustrated as follows.
By considering both the signal and the quantization noise, the power spectra of $x[n]$ and $e[n]$ in the frequency domain are illustrated as

$$\frac{1}{T} = \frac{\Omega NM}{\pi}$$

$$\Phi_{xx}(e^{j\omega})$$

$$\Phi_{ee}(e^{j\omega}) = \sigma_e^2$$
Then, by ideal low pass with cutoff $w_c = \pi/M$ in the decimation, the noise power at the output becomes

$$
\mathbb{E}\{e^2[n]\} = \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \sigma_e^2 \, dw = \frac{\sigma_e^2}{M}
$$
Powers after downsampling

- Next, the lowpass filtered signal is downsampled, and as we have seen, the signal power remains the same. Hence, the power spectrum of $x_{da}[n]$ and $x_{de}[n]$ in the frequency domain are illustrated as follows:

\[
\frac{1}{T'} = \frac{\Omega_N}{\pi}
\]

\[
\Phi_{x_{da}x_{da}}(e^{j\omega})
\]

\[
\Phi_{ee}(e^{j\omega}) = \sigma_e^2 / M
\]
Noise power reduction

- **Conclusion**: The quantization-noise power $\mathcal{E}\{x_{de}^2[n]\}$ has been reduced by a factor of $M$ through the decimation (low-pass filtering + downsampling), while the signal power has remained the same.

\[
\mathcal{E}\{x_{de}^2\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma_e^2}{M} d\omega = \frac{\sigma_e^2}{M} = \frac{\Delta^2}{12M}
\]

- For a given quantization noise power, there is a clear tradeoff between the oversampling factor $M$ and the quantization step $\Delta$. 
Oversampling for noise power reduction

- Remember that \[ \Delta = \frac{X^m}{2^B} \]

- Therefore \[ \mathcal{E}\{x_{de}^2\} = \frac{1}{12M} \left( \frac{X^m}{2^B} \right)^2 \]

- The above equation shows that for a fixed quantizer, the noise power can be decreased by increasing the oversampling ratio M.

- Since the signal power is independent of M, increasing M will increase the signal-to-quantization-noise ratio.
Tradeoff between oversampling and quantization bits

- Alternatively, for a fixed quantization noise power,

\[ P_{de} = \varepsilon \{ x_{de}^2 \} = \frac{1}{12M} \left( \frac{X}{2^B} \right)^2 \]

the required value for \( B \) is

\[ B = -\frac{1}{2} \log_2 M - \frac{1}{2} \log_2 12 - \frac{1}{2} \log_2 P_{de} + \log_2 X_m \]

- From the equation, every doubling of the oversampling ratio \( M \), we need \( \frac{1}{2} \) bit less to achieve a given signal-to-quantization-noise ratio.

- In other words, if we oversample by a factor \( M=4 \), we need one less bit to achieve a desired accuracy in representing the signal.