Homework2

1. (Prob. 3.1 of [1])

Markov's inequality and Chebyshev's inequality

(a) (*Markov's inequality*) For any nonnegative random variable X and any $t > 0$, show that

$$\Pr\{X \geq t\} \leq \frac{EX}{t}. \tag{3.31}$$

Exhibit a random variable that achieves this inequality with equality.

(b) (*Chebyshev's inequality*) Let Y be a random variable with mean μ and variance σ^2. By letting $X = (Y - \mu)^2$, show that for any $\epsilon > 0$,

$$\Pr\{|Y - \mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}. \tag{3.32}$$

(c) (*Weak law of large numbers*) Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Let $\overline{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i$ be the sample mean. Show that

$$\Pr\left\{\left|\overline{Z}_n - \mu\right| > \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}. \tag{3.33}$$

Thus, $\Pr\left\{\left|\overline{Z}_n - \mu\right| > \epsilon\right\} \to 0$ as $n \to \infty$. This is known as the *weak law of large numbers*.

2. (Prob. 3.4 of [1])

AEP. Let X_i be iid $\sim p(x)$, $x \in \{1, 2, \ldots, m\}$. Let $\mu = EX$ and $H = -\sum p(x) \log p(x)$. Let $A^n = \{x^n \in \mathcal{X}^n : |-\frac{1}{n} \log p(x^n) - H| \leq \epsilon\}$. Let $B^n = \{x^n \in \mathcal{X}^n : |\frac{1}{n} \sum_{i=1}^{n} X_i - \mu| \leq \epsilon\}$.

(a) Does $\Pr\{X^n \in A^n\} \longrightarrow 1$?

(b) Does $\Pr\{X^n \in A^n \cap B^n\} \longrightarrow 1$?

(c) Show that $|A^n \cap B^n| \leq 2^{n(H+\epsilon)}$ for all n.

(d) Show that $|A^n \cap B^n| \geq \left(\frac{1}{2}\right) 2^{n(H-\epsilon)}$ for n sufficiently large.

3. (Prob. 3.10 of [1])

Random box size.

An n-dimensional rectangular box with sides $X_1, X_2, X_3, \ldots, X_n$ is to be constructed. The volume is $V_n = \prod_{i=1}^{n} X_i$. The edge length l of a n-cube with the same volume as the random box is $l = V_n^{1/n}$. Let $X_1, X_2, \ldots$ be i.i.d. uniform random variables over the unit interval [0, 1]. Find $\lim_{n\to\infty} V_n^{1/n}$ and compare to $(EV_n)^{\frac{1}{n}}$. Clearly, the expected edge length does not capture the idea of the volume of the box. The geometric mean, rather than the arithmetic mean, characterizes the behavior of products.

4. (Prob. 3.11 of [1])

Proof of Theorem 3.3.1. This problem shows that the size of the smallest "probable" set is about 2^{nH}. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\sim p(x)$. Let $B_\delta^{(n)} \subset \mathcal{X}^n$ such that $\Pr(B_\delta^{(n)}) > 1 - \delta$. Fix $\epsilon < \frac{1}{2}$.

(a) Given any two sets A, B such that $\Pr(A) > 1 - \epsilon_1$ and $\Pr(B) > 1 - \epsilon_2$, show that $\Pr(A \cap B) > 1 - \epsilon_1 - \epsilon_2$. Hence, $\Pr(A_\epsilon^{(n)} \cap B_\delta^{(n)}) \geq 1 - \epsilon - \delta$.

(b) Justify the steps in the chain of inequalities

$$1 - \epsilon - \delta \leq \Pr(A_\epsilon^{(n)} \cap B_\delta^{(n)}) \tag{3.34}$$

$$= \sum_{A_\epsilon^{(n)} \cap B_\delta^{(n)}} p(x^n) \tag{3.35}$$

$$\leq \sum_{A_\epsilon^{(n)} \cap B_\delta^{(n)}} 2^{-n(H-\epsilon)} \tag{3.36}$$

$$= |A_\epsilon^{(n)} \cap B_\delta^{(n)}| 2^{-n(H-\epsilon)} \tag{3.37}$$

$$\leq |B_\delta^{(n)}| 2^{-n(H-\epsilon)}. \tag{3.38}$$

(c) Complete the proof of the theorem.

Let $X_1, X_2, \ldots, X_n$ be i.i.d$\sim p(x)$.

For $\delta < \dfrac{1}{2}$ and any $\delta' > 0$, *if* $\Pr\{B_\delta\} > 1 - \delta$, then

$\dfrac{1}{n} \log |B_\delta{}^{(n)}| > H - \delta'$ for n sufficiently large.

5. (Prob. 4.6 of [1])

Monotonicity of entropy per element. For a stationary stochastic process $X_1, X_2, \ldots, X_n$, show that

(a)

$$\frac{H(X_1, X_2, \ldots, X_n)}{n} \leq \frac{H(X_1, X_2, \ldots, X_{n-1})}{n-1}. \tag{4.94}$$

(b)

$$\frac{H(X_1, X_2, \ldots, X_n)}{n} \geq H(X_n|X_{n-1}, \ldots, X_1). \tag{4.95}$$

6. (Prob. 4.8 of [1])

Maximum entropy process. A discrete memoryless source has the alphabet $\{1, 2\}$, where the symbol 1 has duration 1 and the symbol 2 has duration 2. The probabilities of 1 and 2 are p_1 and p_2, respectively. Find the value of p_1 that maximizes the source entropy per unit time $H(\mathcal{X}) = \frac{H(X)}{ET}$. What is the maximum value $H(\mathcal{X})$?

7. (Prob. 4.12 of [1])

Entropy rate of a dog looking for a bone. A dog walks on the integers, possibly reversing direction at each step with probability $p = 0.1$. Let $X_0 = 0$. The first step is equally likely to be positive or negative. A typical walk might look like this:

$$(X_0, X_1, \ldots) = (0, -1, -2, -3, -4, -3, -2, -1, 0, 1, \ldots).$$

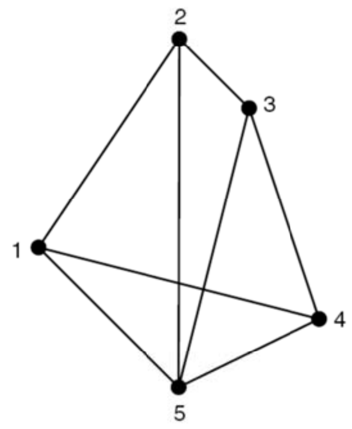
(a) Find $H(X_1, X_2, \ldots, X_n)$.

(b) Find the entropy rate of the dog.

(c) What is the expected number of steps that the dog takes before reversing direction?

8. (Prob. 4.19 of [1])

Random walk on graph. Consider a random walk on the following graph:



(a) Calculate the stationary distribution.

(b) What is the entropy rate?

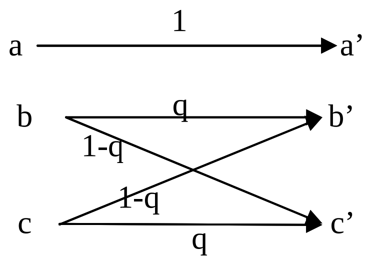
(c) Find the mutual information $I(X_{n+1}; X_n)$ assuming that the process is stationary.

9. Please find the channel capacity of the following channel:

$P(a) = \bar{P}$

$P(b) = P(c) = \bar{Q}$

Channel Capacity C=?



10. For the following BSC

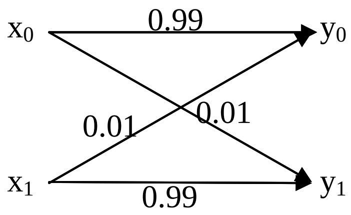
If $P(x_0) = P(x_1) = \frac{1}{2}$,

Please calculate:

(a) $P(y_0)$ and $P(y_1)$

(b) H(Y)

(c) I(X;Y)



11. (Prob. 8.7 of [1])

Differential entropy bound on discrete entropy. Let X be a discrete random variable on the set $\mathcal{X} = \{a_1, a_2, \ldots\}$ with $\Pr(X = a_i) = p_i$. Show that

$$H(p_1, p_2, \ldots) \le \frac{1}{2}\log(2\pi e)\left(\sum_{i=1}^{\infty} p_i i^2 - \left(\sum_{i=1}^{\infty} i p_i\right)^2 + \frac{1}{12}\right). \tag{8.94}$$

Moreover, for every permutation σ,

$$H(p_1, p_2, \ldots) \le \frac{1}{2}\log(2\pi e)\left(\sum_{i=1}^{\infty} p_{\sigma(i)} i^2 - \left(\sum_{i=1}^{\infty} i p_{\sigma(i)}\right)^2 + \frac{1}{12}\right). \tag{8.95}$$

[*Hint:* Construct a random variable X' such that $\Pr(X' = i) = p_i$. Let U be a uniform (0,1] random variable and let $Y = X' + U$, where X' and U are independent. Use the maximum entropy bound on Y to obtain the bounds in the problem. This bound is due to Massey (unpublished) and Willems (unpublished).]

[1] Thomas M. Cover, "*Elements of Information Theory*", Second Edition, 2006