ITCT Lecture 9.1:
Image Data Compression (1)

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Contents

Image Data Compression

I. Introduction :

- Image data Compression is concerned with minimizing the number of bits required to represent an image.
- Applications of data compression are primarily in "Transmission" and "Storage" of information.
- Application of data compression is also in the development of "fast algorithms" where the number of operations required to implement an algorithm is reduced by working with the compressed data.

  --- Compressed Domain Signal Processing

Image data Compression techniques

Pixel Coding

- PCM/quantization
- Run-length coding
- Bit-plane coding

JPEG2000/DVC

Predictive Coding

- Delta modulation
- Line-by-line DPCM
- 2-D DPCM
- Intra-/Inter-frame techniques
- Adaptive

Transform Coding

- Zonal coding
- Threshold coding
- DCT/Waveform
- Real/integer/Lapped
- Multi-D techniques
- Adaptive

Others

- Hybrid Coding
- Vector quantization
- Compressed sensing

- Image data Compression methods fall into two common categories :

A. Redundancy Coding :
   - Redundancy reduction
   - Information lossless

   Predictive coding : DM, DPCM

B. Entropy Coding :
   - Entropy reduction
   - Inevitably results in some distortion

   Transform coding

- For digitized data, “Distortionless Compression” techniques are possible.

Some methods for Entropy reduction:

- Subsampling : reduce the sampling rate
- Coarse Quantization : reduce the number of quantization levels
- Frame Repetition / Interlacing :

  reduce the refresh rate (number of frames per second) TV signals

II. Predictive Techniques :

Basic Principle :

: to remove mutual redundancy between successive pixels and encode only the new information.

DPCM :

A Sampled sequence u(m), coded up to m=n-1. Let $\tilde{u}(n-1), \tilde{u}(n-2), \cdots$ be the value of the reproduced (decoded) sequence.

At m=n, when u(n) arrives, a quantity $\bar{\tilde{u}}(n)$, an estimate of u(n), is predicted from the previously decoded samples $\tilde{u}(n-1), \tilde{u}(n-2), \cdots$, i.e.,

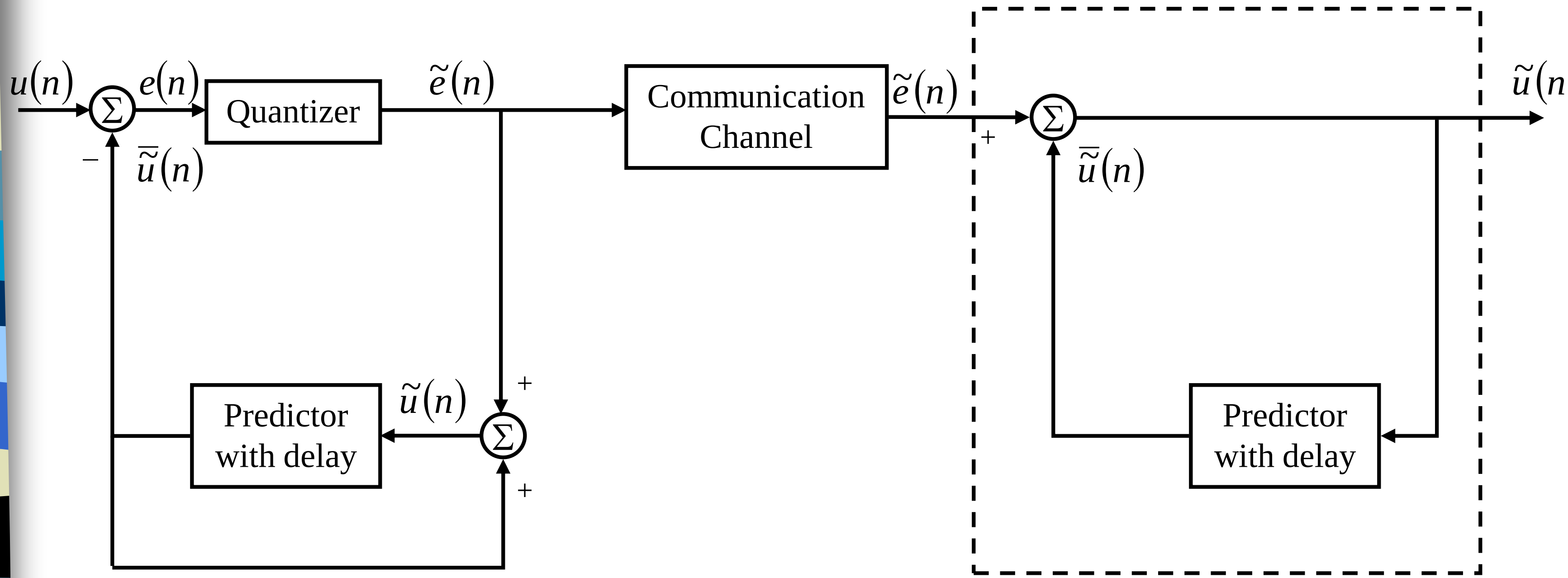
$$\bar{\tilde{U}}(n) = \psi(\tilde{u}(n-1), \tilde{u}(n-2), \cdots) \; ; \; \psi(\cdot) : \text{"prediction rule"}$$

prediction error : $e(n) \overset{\Delta}{=} u(n) - \bar{\tilde{u}}(n)$

If $\tilde{e}(n)$ is the quantized value of e(n), then the reproduced value of u(n) is :

$$\tilde{u}(n) = \bar{\tilde{u}}(n) + \tilde{e}(n)$$

DPCM CODEC



Encoder

Reconstruction
filter/Decoder

- Note :

$$u(n) = \bar{\tilde{u}}(n) + e(n)$$

$$u(n) - \tilde{u}(n) \overset{\Delta}{=} \delta u(n)$$

$$= \left(\bar{\tilde{u}}(n) + e(n)\right) - \left(\bar{\tilde{u}}(n) + \tilde{e}(n)\right)$$

$$= e(n) - \tilde{e}(n)$$

$$= q(n) \qquad : \quad \text{the Quantization error in e(n)}$$

- Remarks:
  1. The pointwise coding error in the input sequence is exactly equal to q(n), the quantization error in e(n)
  2. With a reasonable predictor the mean square value of the differential signal e(n) is much smaller than that of u(n)

- Conclusion:

For the same mean square quantization error, e(n) requires fewer quantization bits than u(n).

$\Rightarrow$ The number of bits required for transmission has been reduced while the quantization error is kept the same.

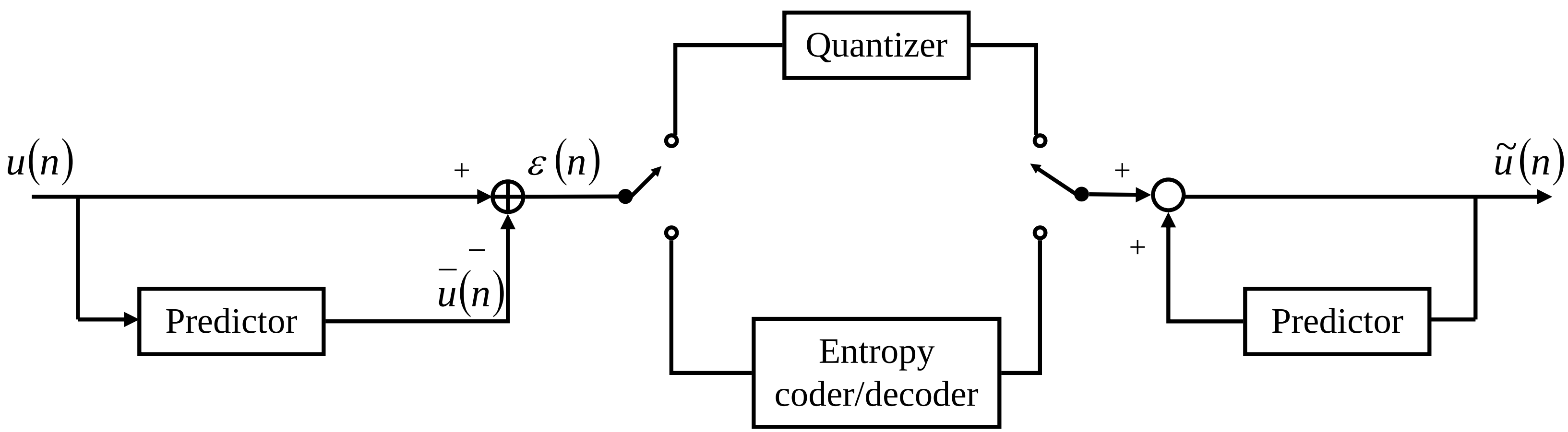
Feedback Versus Feedforward Prediction

An important aspect of DPCM is that the prediction is based on the output — the quantized samples — rather than the input — the unquantized samples.

This results in the predictor being in the "feedback loop" around the quantizer, so that the quantization error at a given step is fed back to the quantizer input at the next step. This has a "stabling effect" that prevents DC drift and accumulation of error in the reconstructed signal $\tilde{u}(n)$.

If the prediction rule is based on the past input, the signal reconstruction error would depend on all the past and present quantization errors in the feedforward prediction-error sequence ε(n).

Generally, the MSE of feedforward reconstruction will be greater than that in DPCM.



Feedforward coding

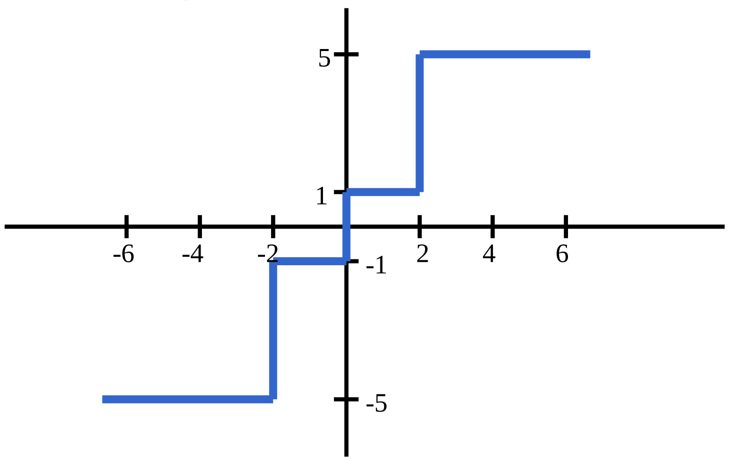
- Example

The sequence 100, 102, 120, 120, 120, 118, 116, is to be predictively coded using the prediction rule:

$$\tilde{\tilde{u}}(n)=\tilde{u}(n-1) \quad \text{for DPCM}$$

$$\bar{u}(n)=u(n-1) \quad \text{for the feedforward predictive coder.}$$

Assume a 2-bit quantizer, as shown below, is used,



Except the first sample is quantized separately by a 7-bit uniform quantizer, given $\tilde{u}(0)=u(0)=100.$

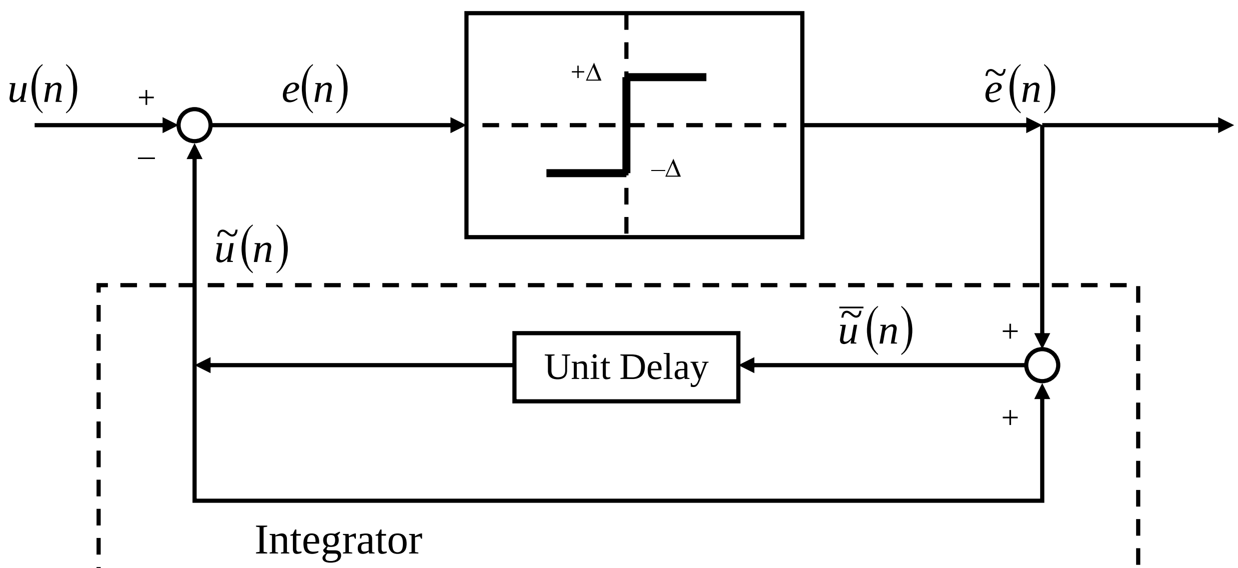
Input		DPCM					Feedforward Predictive Coder				
N	u(n)	$\bar{\tilde{u}}(n)$	$e(n)$	$\tilde{e}(n)$	$\tilde{u}(n)$	δu(n)	$\bar{u}(n)$	$\varepsilon(n)$	$\tilde{\varepsilon}(n)$	$\tilde{u}(n)$	δu(n)
0	100	—	—	—	100	0	—	—	—	100	0
1	102	100	2	1	101	1	100	2	1	101	1
2	120	101	19	5	106	14	102	18	5	106	14
3	120	106	14	5	111	9	120	0	-1	105	15
4	120	111	9	5	116	4	120	0	-1	104	16
5	118	116	2	1	117	1	120	-2	-5	99	19

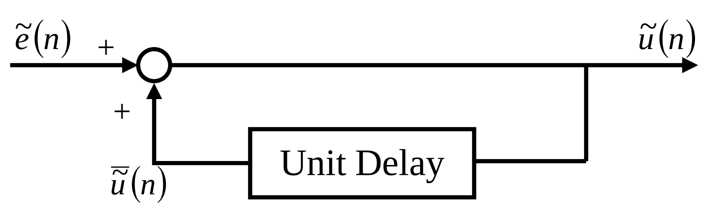
Delta Modulation : (DM)

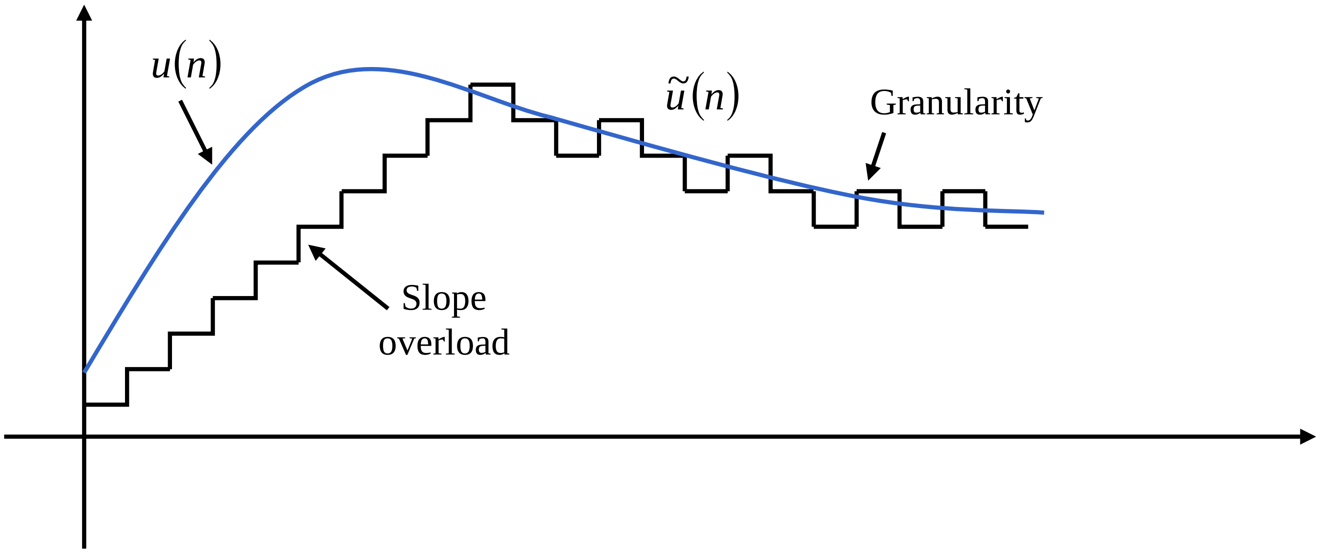
- Predictor : one-step delay function
- Quantizer : 1-bit quantizer

$$\tilde{\tilde{u}}(n) = \tilde{u}(n-1)$$

$$e(n) = u(n) - \tilde{u}(n-1)$$







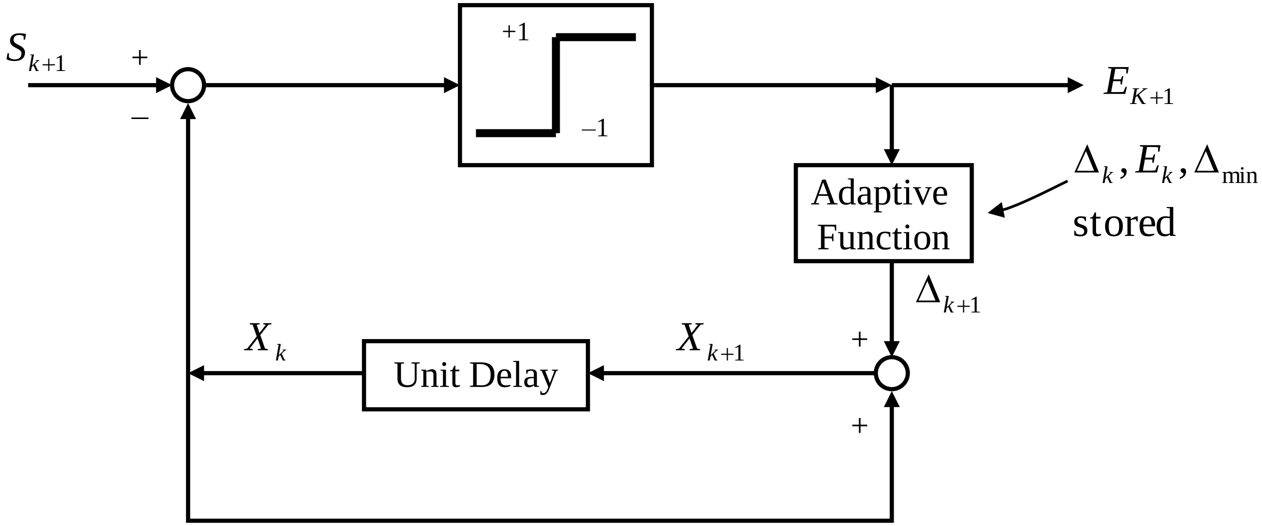
- Primary Limitation of DM :

1) Slope overload : large jump region
   Max. slope = (step size) × (sampling freq.)
2) Granularity Noise : almost constant region
3) Instability to channel Noise

Step size effect :

Step Size ↑ ⇒ (i) slope overload ↓
(sampling frequency ↑) (ii) granular Noise ↑

Adaptive Delta Modulation



$$E_{K+1} = \operatorname{sgn}\left[S_{K+1} - X_K\right]$$

$$\Delta_{K+1} = \begin{Bmatrix} \left|\Delta_K\right|\left[E_{K+1} - \frac{1}{2}E_K\right] & \text{if } \left|\Delta_K\right| \geq \Delta_{\min} \\ \Delta_{\min} E_{K+1} & \text{if } \left|\Delta_K\right| < \Delta_{\min} \end{Bmatrix}$$

$$X_{K+1} = X_K + \Delta_{K+1}$$

This adaptive approach simultaneously minimizes the effects of both slope overload and granular noise.

DPCM Design

- There are two components to design in a DPCM system :
  i. The predictor
  ii. The quantizer

Ideally, the predictor and quantizer would be optimized together using a linear or Nonlinear technique. In practice, a suboptimum design approach is adopted :

i. Linear predictor
ii. Zero-memory quantizer

Remark : For this approach, the number of quantizing levels, M, must be relatively large ($M \geq 8$) to achieve good performance.

Design of linear predictor

$$\hat{S}_0 = a_1 S_1 + a_2 S_2 + \cdots + a_n S_n$$

$$e_0 = S_0 - \hat{S}_0$$

$$\frac{\partial E\left[\left(S_0 - \hat{S}_0\right)^2\right]}{\partial a_i} = \frac{\partial E\left[\left(S_0 - \left(a_1 S_1 + a_2 S_2 + \cdots + a_n S_n\right)\right)^2\right]}{\partial a_i}$$

$$= -2E\left[\left(S_0 - \left(a_1 S_1 + a_2 S_2 + \cdots + a_n S_n\right)\right) S_i\right]$$

$$= 0 \quad , \quad \mathrm{i} = 1,2,\cdots \mathrm{n}$$

$$\Rightarrow E\left[\left(S_0 - \left(a_1 S_1 + a_2 S_2 + \cdots + a_n S_n\right)\right) S_i\right] = 0$$

$$E\left[\left(S_0 - \hat{S}_0\right) S_i\right] = 0, \quad \mathrm{i} = 1,2,\cdots \mathrm{n}$$

$$R_{ij} = E\left[S_i S_j\right]$$

$$E\left[S_0 S_i\right] = E\left[\hat{S}_0 S_i\right]$$

$$R_{0i} = E\left[a_1 S_1 S_i + a_2 S_2 S_i + \cdots + a_n S_n S_i\right]$$

$$= a_1 R_{1i} + a_2 R_{2i} + \cdots + a_n R_{ni}$$

$$\left[R_{0i}\right] = \left[R_{1i}, R_{2i}, \cdots, R_{ni}\right] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\left[a_i\right] = \left[R_{1i}, R_{2i}, \cdots, R_{ni}\right]^{-1} \left[R_{0i}\right]$$

- When $\hat{S}_0$ comprises these optimized coefficients, a_i, then the mean square error signal is :

$$\sigma_e^2 = E\left[\left(S_0 - \hat{S}_0\right)^2\right]$$
$$= E\left[\left(S_0 - \hat{S}_0\right)S_0\right] - E\left[\left(S_0 - \hat{S}_0\right)\hat{S}_0\right]$$

But $E\left[\left(S_0 - \hat{S}_0\right)\hat{S}_0\right] = 0$ (orthogonal principle)

$$\sigma_e^2 = E\left[\left(S_0 - \hat{S}_0\right)S_0\right] = E\left[S_0^2\right] - E\left[\hat{S}_0 S_0\right]$$
$$= R_{00} - \left(a_1 R_{01} + a_2 R_{02} + \cdots + a_n R_{0n}\right)$$

σ_e^2 : the variance of the difference signal

R_{00} : the variance of the original signal

$\Rightarrow$ The variance of the error signal is less than the variance of the original signal.

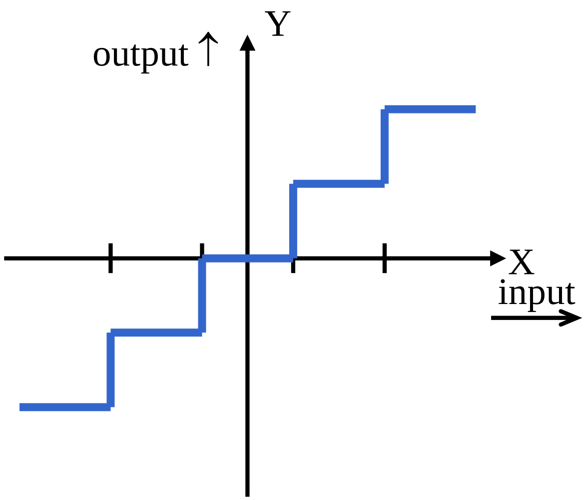
- Remarks:

1. The complexity of the predictor depends on “n”.
2. “n” depends on the covariance properties of the original signal.

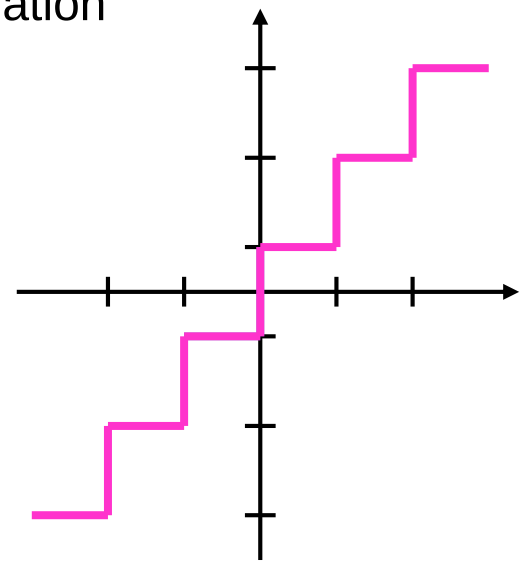
Design of the DPCM Quantizer

- Review of uniform Quantizer:

Quantization: 1. Zero-Memory quantization
2. Block quantization
3. Sequential quantization



Midtread quantizer



Midriser quantizer

Quantization Error : $Q_E = y(X) - X$

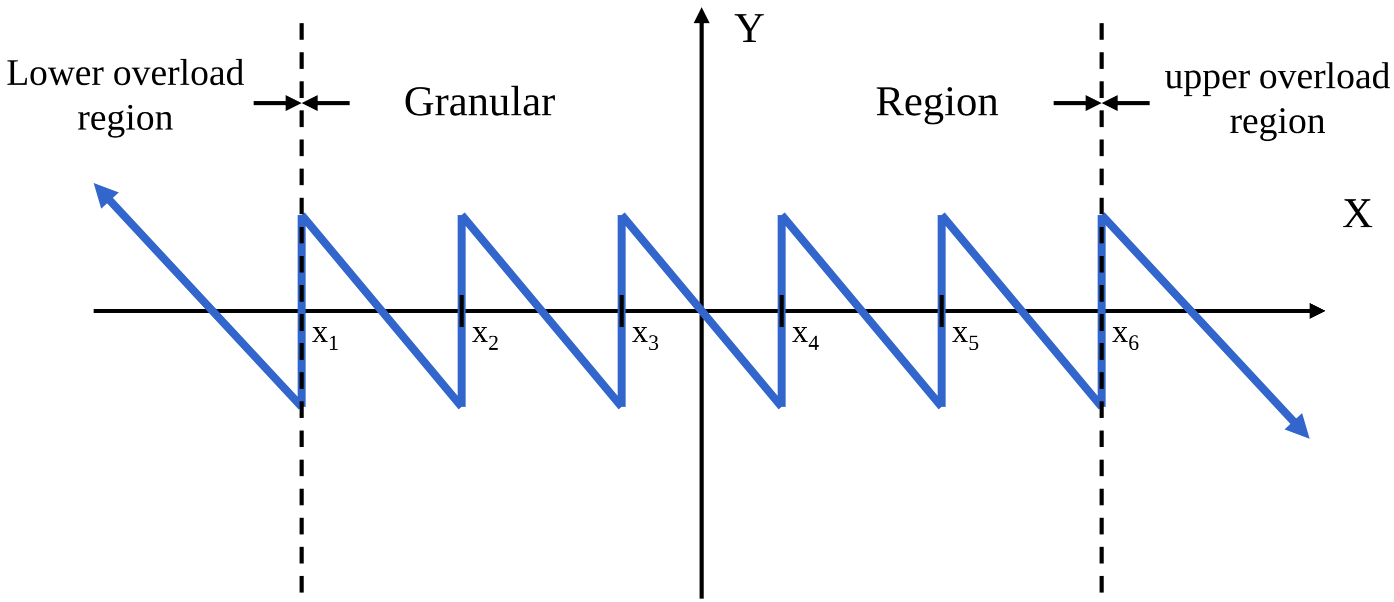
Average Distortion : $D = \int_{-\infty}^{\infty} [y(X) - X]^2 P(X) dX$

SNR :

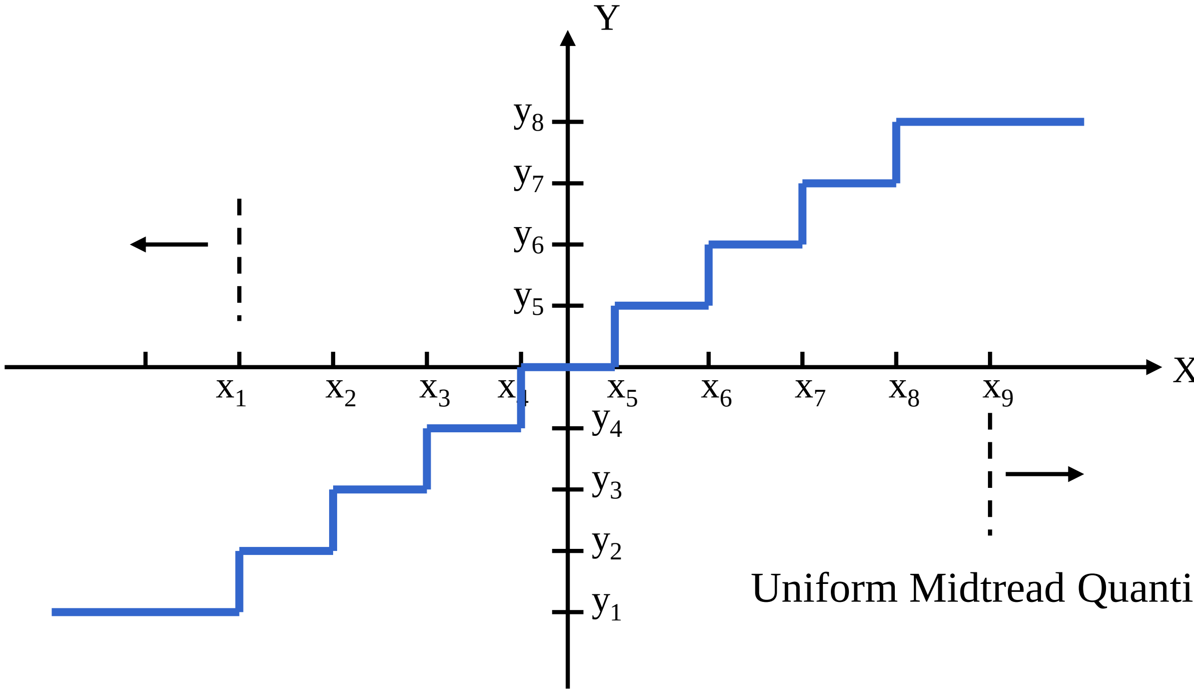
$$SNR = 10\log_{10}\left(\frac{\sigma^2}{D}\right) \text{ in dB}$$

where σ^2 : the variance of the input x

- Uniform quantizer : p(x) is constant within each interval



Quantization Error for Midtread Quantizer



Uniform Midtread Quantizer M=9

- Output level y_i always be in the midpoint of the input interval $\Delta=x_i-x_{i-1}$. Assume p(x) is constant in the interval $\Delta=x_i-x_{i-1}$ and equal to $p(x_i)$

  Lower overload region : $\Delta=x_0-x_1$, $x_1 >> x_0$
  Granular region : $\Delta=x_i-x_{i-1}$, $2 \le i \le M-1$
  upper overload region : $\Delta=x_M-x_{M-1}$, $x_M >> x_{M-1}$

$$D=\sum_{i=1}^{M}\int_{x_{i-1}}^{x_i}\left[y_i(x)-x\right]^2 p(x_i)\,dx$$

$$\cong \sum_{i=2}^{M-1} p(x_i)\left\{\frac{\left[y_i(x)-x\right]^3}{3}\right\}\Bigg|_{x_{i-1}}^{x_i}$$

(Where we assume the contribution of the overload region is negligible ; i.e. $p(x_1)=p(x_M)=0$)

Since $x_i = y_i(x) + \frac{\Delta}{2}$

$x_{i-1} = y_i(x) - \frac{\Delta}{2}$

(Quantizer characteristics)

$$\Rightarrow D \cong \frac{1}{12} \sum_{i=2}^{M-1} p(x_i)\Delta^3$$

But $\sum_{i=2}^{M-1} p(x_i)\Delta \approx 1$

$$\Rightarrow D \cong \frac{\Delta^2}{12}$$

(Source Model)

if the pdf is $p(x) = \frac{1}{2V} \quad (-V \le x \le V)$

the input variance is

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 p(x)$$

$$= \int_{-V}^{V} x^2 \cdot \frac{1}{2V}\, dx = \frac{V^2}{3}$$

Then

$$SNR = 10\log_{10}\frac{\sigma^2}{D}$$

$$= 10\log_{10}\frac{V^2 \cdot 12}{3\cdot\Delta^2}$$

But $\Delta = \frac{2V}{M}$ for $M \geq 2$

$$\Rightarrow SNR \cong 10\log_{10} M^2 = 20\log_{10} M$$

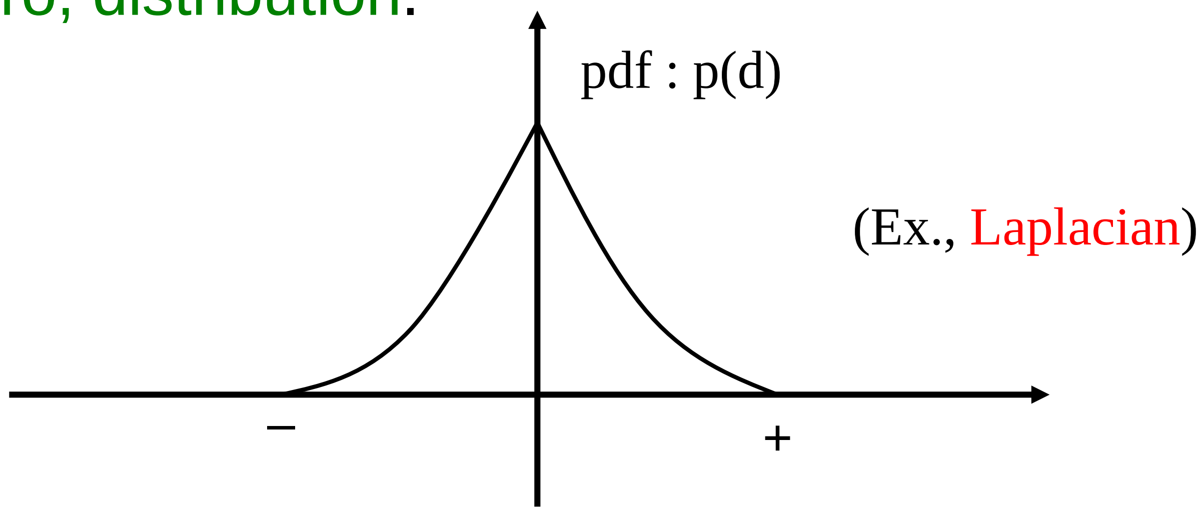
if $M = 2^n$ (n - bit quantizer)

$$SNR \cong 20n\log_{10} 2 = 6n \text{ (in dB)}$$

- valid only for PCM Quantizer

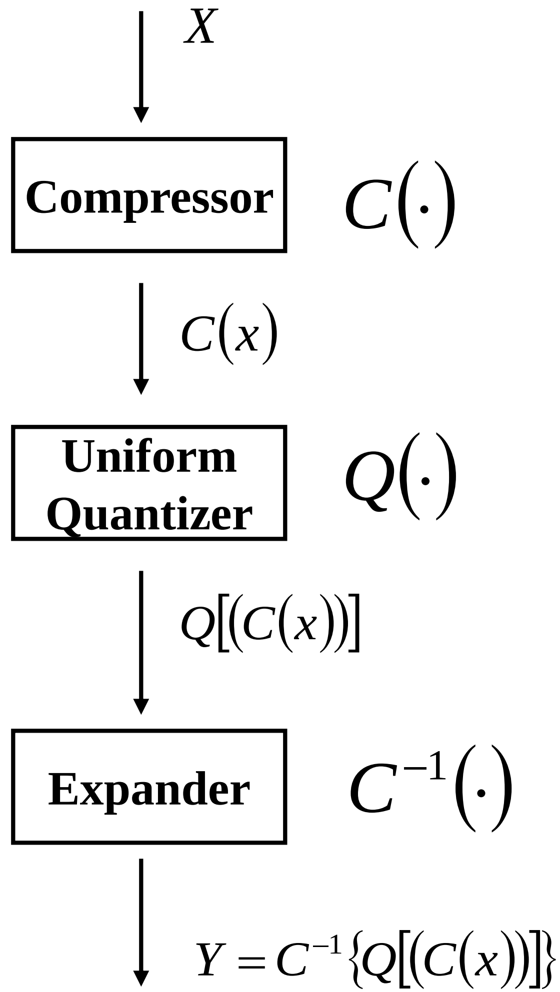
B. DPCM Quantizer

The pdf of the input signal to the DPCM quantizer is not at all uniform. Since a "good" predictor would be expected to result in many zero difference between the predicted values and the actual values. A typical shape for this distribution is a highly peaked, around zero, distribution.

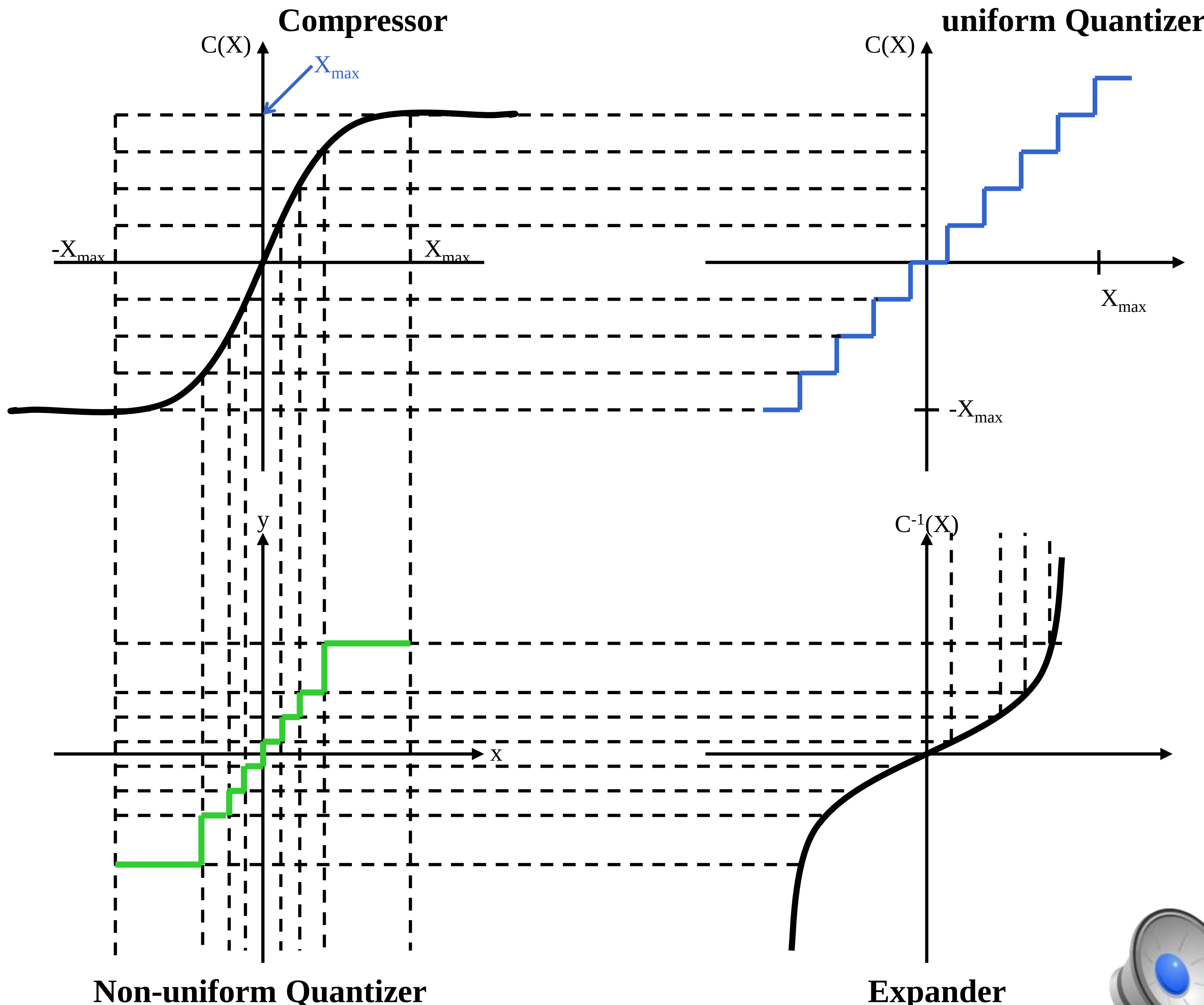


: Non-uniform Quantizer is required.

$$\frac{dC(x)}{dx} \approx \frac{2x_{\max}}{L\Delta_x}$$



Non-uniform Quantizer ↔ compressor + uniform Quantizer + Expander



Compressor
uniform Quantizer
C(X)
C(X)
Xmax
-Xmax
Xmax
Xmax
-Xmax
y
x
C-1(X)
Non-uniform Quantizer
Expander

For this model, the mean-square distortion can be approximately represented as :

$$D = \frac{1}{12M^2} \int_{L_1}^{L_2} \frac{p(x)}{[\lambda(x)]^2} \cdot dx$$

where

$$\lambda(x) = C'(x) \big/ (L_2 - L_1)$$

$L_2 - L_1$ is the quantizer range

$C'(x)$ is the slope of the nonlinear function

Lloyd-Max Quantizer : the most popular one.

1. Each interval limit should be midway between the neighboring levels,

$$x_i = \frac{(y_i + y_{i+1})}{2}$$

2. Each level should be at the centroid of the input prob. Density function over the interval for that level, that is

$$\int_{x_{i-1}}^{x_i} (x - y_i)\, p(x)\, dx = 0$$

Logarithmic Quantizer : ⟶ (log PCM)

$\frac{dC(x)}{dx} = (KX)^{-1}$

μ-law

$$y(x) = \frac{V \log\left(1 + {}^{1+\mu x}\!/_{V}\right)}{\log(1+\mu)}$$

: US. Canada, Japan

A-law

$$y(x) = \begin{cases} \frac{Ax}{1+\log A} & , 0 \le x \le \frac{V}{A} \\ \frac{V + V\log\left({}^{Ax}\!/_{V}\right)}{1+\log A} & , \frac{V}{A} \le x \le V \end{cases}$$

: Europe

- If a Laplacian function is used to model p(e),

$$p(e)=\frac{1}{\sqrt{2}\sigma_e}\exp\left(-\frac{\sqrt{2}}{\sigma_e}|e|\right)$$

Input pdf of the DPCM Quantizer

then the variance of the quantization error is:

$$\sigma_g^2=\frac{2}{3M^2}\left[\int_0^V \frac{1}{\left(\sqrt{2}\sigma_e\right)^{1/3}}\exp\left(\frac{-\sqrt{2}}{3\sigma_e}|e|\right)\,de\right]^3$$

$$\sigma_g^2 \cong \frac{9\sigma_e^2}{2M^2} \qquad \text{as } V\to\infty$$

$\Rightarrow$ the SNR for the non-uniform quantizer in DPCM becomes :

$$SNR=10\log_{10}\left(\frac{\sigma^2}{\sigma_g^2}\right)$$

$$\cong 10\log_{10}\left(\frac{2M^2\sigma^2}{9\sigma_e^2}\right)$$

Since $M=2^n$

$$SNR\cong -6.5+6n+10\log_{10}\frac{\sigma^2}{\sigma_e^2}$$

For the same pdf, PCM gives :

$$SNR\cong -6.5+6n$$

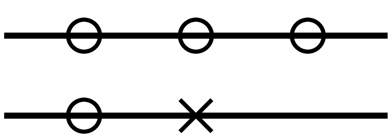
$\Rightarrow$ DPCM improves the SNR by

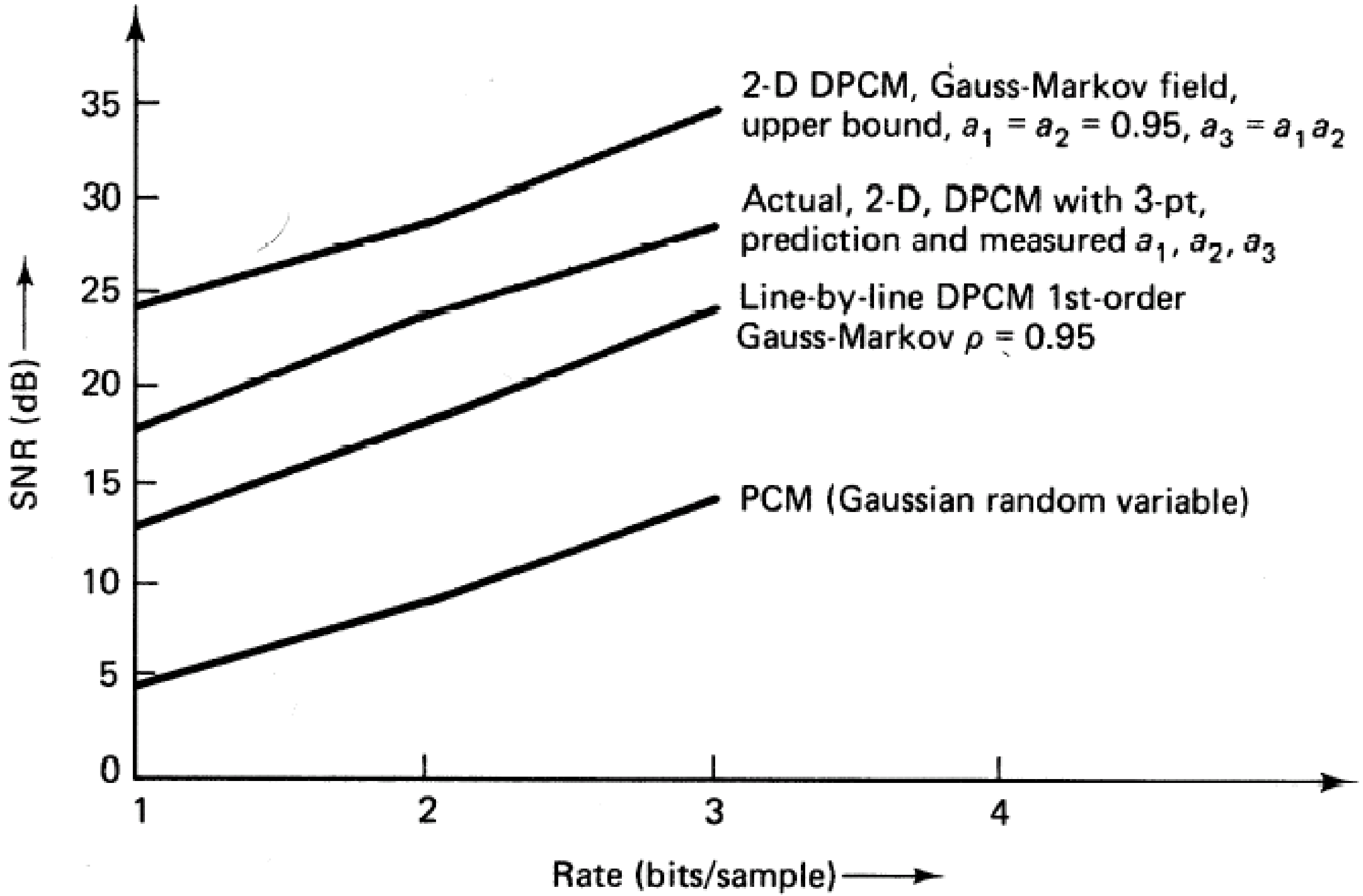
$$10\log_{10}\frac{\sigma^2}{\sigma_e^2}$$

- ADPCM :
  i. Adaptive prediction
  ii. Adaptive Quantization
- DPCM for Image Coding :
  Each scan line of the image is coded independently by the DPCM techniques. For every slow time-varying image (ρ=0.95) and a Laplacian-pdf Quantizer,
    8 to 10 dB SNR improvement over PCM can be expected : that is
  The SNR of 6-bit PCM can be achieved by 4-bit line-by-line DPCM for ρ=0.97.
- Two-Dimensional DPCM : two-D predictor

Ex :

$$\bar{u}(m,n) = a, u(m-1,n) + a_2 u(m,n-1)$$
$$a_3 u(m-1,n-1) + a_4 u(m-1,n+1)$$





Performance of predictive codes.