定理 1  
(1) \( \langle Ax, y \rangle = \langle x, A^* y \rangle \) 
(2) \( A^{**} = A \), \( (\lambda A)^* = \overline{\lambda} A^* \) 
(3) \( A\) 可逆 \( \Rightarrow A^* \) 可逆, \( (A^*)^{-1} = (A^{-1})^* \) 

定理 2  
(1) \( (e^A)^* = e^{A^*} \) 
(2) \( AB = BA \Rightarrow e^A e^B = e^{A+B} \) 

定理 3  
\( A x = \lambda x \Rightarrow \) (1) \( A^{-1} x = \lambda^{-1} x \) 
(2) \( e^{A} x = e^{\lambda} x \) \((\sum_{k=0}^{\infty} \frac{A^k}{k!} x = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} x = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{A^k B^{n-k}}{k! (n-k)!} x)\) 

定理 4  
\( A : \) Hermitian \( \Rightarrow (o) A^{-1} : \) Hermitian 
(1) \( \forall \) eigen values \( \in IR \), \( \{ A x = \lambda x, A y = \mu y \} \lambda \neq \mu \Rightarrow x \perp y \) 
(2) \( \exists \) orthonormal eigen basis \( \{ u_1, ..., u_n \} \) 
\( A = [u_1 ... u_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m \end{bmatrix} [u_1^* ... u_n^*] = \lambda_1 u_1 u_1^* + \cdots + \lambda_m u_m u_m^* \) 
(\( u \perp u^* \)) 
(3) \( A^{-1} : \) Hermitian 
\( A^{-1} = [u_1 ... u_n] \begin{bmatrix} \lambda_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m^{-1} \end{bmatrix} [u_1^* ... u_n^*] = \lambda_1^{-1} u_1 u_1^* + \cdots + \lambda_m^{-1} u_m u_m^* \) 

定理 5  
\( A : \) Hermitian \( \Rightarrow e^{iA} \) unitary \( (e^{iA}, e^{iAt}, e^{-iAt}) \) 

证 \( (e^{iA})^* e^{iA} = e^{-iA} e^{iA} = \theta = I \)
Rotational Operators

\[
\begin{align*}
R_z(\theta) &= e^{i\theta \sigma_z} = e^{i\frac{\theta}{2} \sigma_z} \\
R_x(\theta) &= e^{i\frac{\theta}{2} \sigma_x} \\
R_y(\theta) &= e^{i\frac{\theta}{2} \sigma_y}
\end{align*}
\]

\[
e^{i\frac{\theta}{2} \sigma_j} = \sum_{n=0}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} \sigma_j^n
\]

\[
\begin{align*}
\sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma_z^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma_y^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma_x^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
R_z(\theta) &= e^{i\frac{\theta}{2} \sigma_z} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(-i\frac{\theta}{2})^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \\
R_y(\theta) &= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} & -i \sum_{n=1,3,5,...}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} \\ i \sum_{n=1,3,5,...}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} & \sum_{n=0,2,4,...}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\
R_x(\theta) &= \begin{pmatrix} \sum_{n=0,2,4,...}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} & \sum_{n=1,3,5,...}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} \\ \sum_{n=1,3,5,...}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} & \sum_{n=0,2,4,...}^{\infty} \frac{(i\frac{\theta}{2})^n}{n!} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!}
\end{align*}
\]
HHL Algorithm (Harrow, Hassidim, Lloyd, '09)

Given \( A = \text{Hermitian} \), solve \( A x = b \) \((A = A^*)\)

1. \( A = A^* \), solve
\[
\begin{bmatrix}
0 & A \\
A^* & 0
\end{bmatrix}
\begin{bmatrix}
x \\\n\bar{x}
\end{bmatrix} =
\begin{bmatrix}
b \\
0
\end{bmatrix}
\]

2. \( A = \text{Hermitian} \Rightarrow A^{-1} = \text{Hermitian} \)
   \( A x = \lambda x \Rightarrow A^{-1} x = \frac{1}{\lambda} x \)
   \( A = \begin{bmatrix}
\lambda_1 & \cdots & \lambda_n
\end{bmatrix}_B \Rightarrow A^{-1} = \begin{bmatrix}
\frac{1}{\lambda_1} & \cdots & \frac{1}{\lambda_n}
\end{bmatrix}_B \)
   \( = \frac{1}{\lambda_1} u_1 u_1^* + \cdots + \frac{1}{\lambda_n} u_n u_n^* \)

3. \( b = \beta_1 u_1 + \cdots + \beta_n u_n \)
   \( A x = b \Rightarrow x = A^{-1} b \)
   \( = \begin{bmatrix}
\frac{1}{\lambda_1} & \cdots & 0
\end{bmatrix} \begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_n
\end{bmatrix} \)
   \( = \frac{\beta_1}{\lambda_1} u_1 + \cdots + \frac{\beta_n}{\lambda_n} u_n \)
   \( = \sum_{j=1}^{m} \frac{\beta_j}{\lambda_j} |u_j\rangle \)

4. \( \{ A^{-1} \text{ not unitary, but } e^{iA}, e^{iA^*} \text{ unitary} \) (Hamiltonian simulation)

5. \( \sin \frac{\theta}{2} = \frac{c}{\lambda} \)
   \( R_y(\theta) = e^{i\frac{\theta}{2} Y} = \begin{bmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{bmatrix} : |0\rangle \mapsto \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle \)
   \( = \sqrt{1 - \frac{c^2}{\lambda^2}} |0\rangle + \frac{c}{\lambda} |1\rangle \)
Algorithm 4 HHL algorithm

Input:
- The state $|b\rangle = \sum_j \beta_j |u_j\rangle$
- The ability to perform controlled operations with unitaries of the form $e^{i\Lambda}$

Output:
- The quantum state $|x\rangle$ such that $A|x\rangle = \tilde{b}$.

Procedure:

Step 1. Perform quantum phase estimation using the unitary transformation $e^{iA}$. This maps the eigenvalues $\lambda_j$ into the register in the binary form to transform the system,

$$|0\rangle_a |0\rangle_r |b\rangle_m \rightarrow \sum_{j=1}^{N} \beta_j |0\rangle_a |\lambda_j\rangle_r |u_j\rangle_m .$$

Step 2. Rotate the ancilla qubit $|0\rangle_a$ to $\sqrt{1 - \frac{C^2}{\lambda_j^2}} |0\rangle_a + \frac{C}{\lambda_j} |1\rangle_a$ for each $\lambda_j$. This is performed through controlled rotation on the $|0\rangle_a$ ancilla qubit. The system will evolve to

$$\sum_{j=1}^{N} \beta_j \left( \sqrt{1 - \frac{C^2}{\lambda_j^2}} |0\rangle_a + \frac{C}{\lambda_j} |1\rangle_a \right) |\lambda_j\rangle_r |u_j\rangle_m .$$

Step 3. Perform the reverse of Step 1. This will lead the system to

$$\sum_{j=1}^{N} \beta_j \left( \sqrt{1 - \frac{C^2}{\lambda_j^2}} |0\rangle_a + \frac{C}{\lambda_j} |1\rangle_a \right) |0\rangle_r |u_j\rangle_m .$$

Step 4. Measuring the ancilla qubit will give,

$$|x\rangle \approx \sum_{j=1}^{N} C \left( \frac{\beta_j}{\lambda_j} \right) |u_j\rangle ,$$

if the measurement outcome is $|1\rangle$. (amplitude amplification)
### 例子

\[ A = \frac{1}{2} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}; \hat{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \]

\( \lambda_1 = 1: |x_2x_3⟩ = |01⟩, |u_1⟩ \)

\( \lambda_2 = 2: |x_2x_3⟩ = |10⟩, |u_2⟩ \)

\( |b⟩ = b_1|0⟩ + b_2|1⟩ = \beta_1|u_1⟩ + \beta_2|u_2⟩ \)

#### PE:

\[
\begin{align*}
|x_2x_3x_4⟩ &= \beta_1|01⟩|u_1⟩ + \beta_2|10⟩|u_2⟩ \\
\end{align*}
\]

#### CR&M:

\[
\begin{align*}
|x_4⟩ &= \beta_1|u_1⟩ + \frac{1}{2}\beta_2|u_2⟩ \\
λ_1^{-1}\beta_1|u_1⟩ + λ_2^{-1}\beta_2|u_2⟩ \\
\end{align*}
\]

#### Swap:

\[
\begin{align*}
|x_2x_3x_4⟩ &= \beta_1|10⟩|u_1⟩ + \beta_2|01⟩|u_2⟩ \\
\end{align*}
\]

\[
\begin{align*}
\beta_1|2λ_1^{-1})|u_1⟩ + \beta_2|2λ_2^{-1})|u_2⟩ \\
\end{align*}
\]

\[ A^{-1}b \]

\[
\begin{align*}
\begin{bmatrix}
\frac{1}{2} & 1 \\
1 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} \\
1
\end{bmatrix}
= & \begin{bmatrix}
1 \\
\frac{1}{2}
\end{bmatrix} \quad λ_1 = 1 \quad |u_1⟩ = \begin{bmatrix}
\frac{1}{2} \\
-1
\end{bmatrix} \\
\frac{1}{\sqrt{2}} |1⟩
= & \begin{bmatrix}
\frac{1}{\sqrt{2}} \\
1
\end{bmatrix} \quad λ_2 = 2 \quad |u_2⟩ = \begin{bmatrix}
\frac{1}{\sqrt{2}} \\
1
\end{bmatrix}
\end{align*}
\]
Example: $A_{4 \times 4}$