Analysis of Binary Search Trees (BST) Algorithms

Goal: to derive

$$\begin{cases} \mathbb{E}(X_n) & := \# \text{ probes per } \quad \textbf{succ} \text{ search in } n\text{-node BST} \\ \mathbb{E}(Y_n) & := \# \text{ probes per } \textbf{unsucc} \text{ search} \qquad \text{"} \end{cases}$$

Sample Spaces:

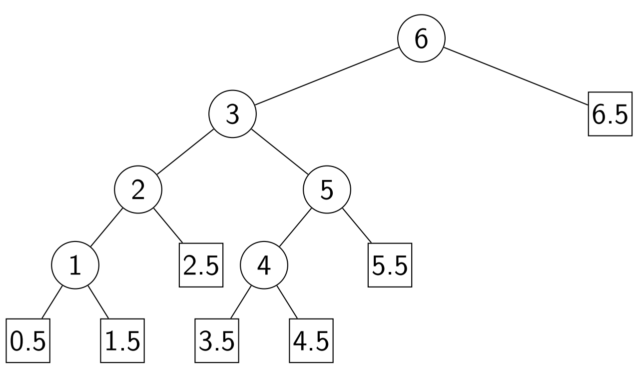
$$\begin{cases} \Omega_{X_n} & = \left\{ (x_1, x_2, \ldots, x_n; k) \,\middle|\, \begin{matrix} (x_1, x_2, \ldots, x_n) \in S_n \\ 1 \le k \le n \end{matrix} \right\}, \quad |\Omega_{X_n}| = n!\, n \\ \Omega_{Y_n} & = \left\{ (x_1, x_2, \ldots, x_n; y) \,\middle|\, \begin{matrix} (x_1, x_2, \ldots, x_n) \in S_n \\ y = 0.5,\ 1.5,\ \ldots,\ n.5 \end{matrix} \right\}, \quad |\Omega_{Y_n}| = (n+1)! \end{cases}$$

$$\begin{aligned} S_n &= \text{ set of all permutations (relative order of keys) of } \{1, 2, \ldots, n\} \\ \Omega_{Y_n} &\approx \big\{ (x_1, x_2, \ldots, x_n; y) \mid (x_1, x_2, \ldots, x_n; y) \in S_{n+1} \big\}, \quad \text{by re-numbering} \end{aligned}$$

Random Variables:

$$\begin{cases} X_n\big((x_1, x_2, \ldots, x_n; k)\big) & := \# \text{ probes to } \quad \textbf{succ} \text{ search } x_k \text{ in } T(x_1, x_2, \ldots, x_n) \\ Y_n\big((x_1, x_2, \ldots, x_n; y)\big) & := \# \text{ probes to } \textbf{unsucc} \text{ search } y \qquad \text{"} \end{cases}$$

$$\begin{aligned} T(x_1, x_2, \ldots, x_n) &:= \text{ BST formed by inserting keys } x_1, x_2, \ldots, x_n \\ T(6, 3, 2, 1, 5, 4) &= \end{aligned}$$



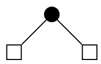
$$\begin{cases} \displaystyle\sum_{1 \le k \le 6} X_6\big((6,3,2,1,5,4;k)\big) & = (1+2+3+4+3+4) \qquad = 17 = 6 + I\big(T(6,3,2,1,5,4)\big) \\ \displaystyle\sum_{y=0.5,\ldots,6.5} Y_6\big((6,3,2,1,5,4;y)\big) & = (4+4+3+4+4+3+1) \quad = 23 = \qquad E\big(T(\cdot)\big) \end{cases}$$

$$\begin{cases} I(T) & := \displaystyle\sum_{x\text{: int node of } T} \text{path length}(x) = 11 \\ E(T) & := \displaystyle\sum_{y\text{: ext node of } T} \text{path length}(y) = 23 \quad = \; I(T) + 2 \cdot 6 \end{cases}$$

$$\mathbb{E}(X_6) = \frac{1}{6}\Big[\big(1 + \mathbb{E}(Y_0)\big) + \big(1 + \mathbb{E}(Y_1)\big) + \cdots + \big(1 + \mathbb{E}(Y_4)\big) + \big(1 + \mathbb{E}(Y_5)\big) \Big]$$

n	1	2	3	
$\mathbb{E}(X_n) = 2\frac{n+1}{n}H_n - 3$	1	$\frac{3}{2}$	$\frac{17}{9}$	$\cdots$
$\mathbb{E}(Y_n) = 2H_{n+1} - 2$	1	$\frac{5}{3}$	$\frac{13}{6}$	$\cdots$

$\boxed{n = 1}$



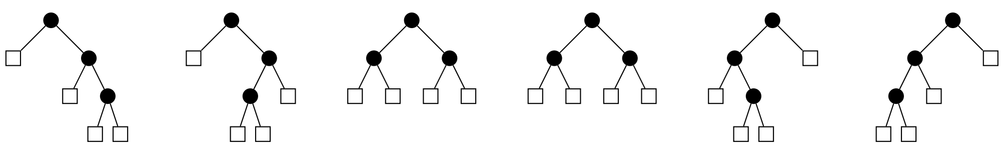
$$\begin{cases} \mathbb{E}(X_1) & = \frac{1}{1}\,[1] & = 1 \\ \mathbb{E}(Y_1) & = \frac{1}{2}\,[1+1] & = 1 \end{cases}$$

$\boxed{n = 2}$

$(1,2)$ $(2,1)$

$$\begin{cases} \mathbb{E}(X_2) & = \frac{1}{2\cdot 2!}\,[(1+2)+(1+2)] & = \frac{3}{2} \\ \mathbb{E}(Y_2) & = \frac{1}{3\cdot 2!}\,[(1+2+2)+(2+2+1)] & = \frac{5}{3} \end{cases}$$

$\boxed{n = 3}$



$(1,2,3)$ $(1,3,2)$ $(2,1,3)$ $(2,3,1)$ $(3,1,2)$ $(3,2,1)$

$$\begin{cases} \mathbb{E}(X_3) & = \frac{1}{3\cdot 3!}\,[(1+2+3)+(1+2+3)+(1+2+2)+(\cdot)+(\cdot)+(\cdot)] & = \frac{17}{9} \\ \mathbb{E}(Y_3) & = \frac{1}{4\cdot 3!}\,[(1+2+3+3)+(1+3+3+2)+(2+2+2+2)+(\cdot)+(\cdot)+(\cdot)] & = \frac{13}{6} \end{cases}$$

Remark:

$$\begin{cases} \textbf{Dynamic}\ \text{model} & : n!\ \text{trees;} \\ \textbf{Static}\ \quad " & : \frac{1}{n+1}\binom{2n}{n}\ \text{trees,} \quad \big[T(2,1,3) = T(2,3,1)\big] \end{cases}$$

Lemma 1. n-node binary tree T has
(1) $2n$ edges,
(2) $n+1$ external nodes (leaves),
(3) $E(T) = I(T) + 2n$.

Lemma 2. $(n+1)\mathbb{E}(Y_n) = n\mathbb{E}(X_n) + n$

Proof:

$$
\begin{aligned}
(1)\ \mathbb{E}(X_n) &= \sum_{w\in\Omega_{X_n}} P(w)X_n(w) = \sum_{\substack{(x_1,\ldots,x_n)\in S_n \\ 1\le k\le n}} \frac{1}{n!\,n} X_n(x_1,\ldots,x_n;k) \\
&= \frac{1}{n!\,n} \sum_{(x_1,\ldots,x_n)\in S_n} \sum_{1\le k\le n} X_n(x_1,\ldots,x_n;k) \\
&= \frac{1}{n!\,n} \sum_T \big(n + I(T)\big) \\
&= \frac{1}{n}\Big[n + \frac{1}{n!}\sum_T I(T)\Big]
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \mathbb{E}(Y_n) &= \frac{1}{(n+1)!} \sum_{(x_1,\ldots,x_n)\in S_n} \sum_{y=0.5,\ldots,n.5} Y_n(x_1,\ldots,x_n;y) \\
&= \frac{1}{(n+1)!} \sum_T E(T) \\
&= \frac{1}{(n+1)n!} \sum_T \big(2n + I(T)\big) \\
&= \frac{1}{n+1}\Big[2n + \frac{1}{n!}\sum_T I(T)\Big]
\end{aligned}
$$

$(1) + (2) \Rightarrow (n+1)\mathbb{E}(Y_n) = n\mathbb{E}(X_n) + n$ ■

Lemma 3. $\mathbb{E}(X_n) = 1 + \frac{1}{n}\sum_{1\le k\le n} \mathbb{E}(Y_{k-1})$

Theorem.

$$
\begin{cases}
\mathbb{E}(X_n) & = 2\frac{n+1}{n}H_n - 3 \quad \approx 2\ln n \\
\mathbb{E}(Y_n) & = 2H_{n+1} - 2 \quad \approx 2\ln n
\end{cases}
$$

Proof: Lemmas 2 and 3 give

$$
\begin{aligned}
(n+1)\,\mathbb{E}(Y_n) &= \sum_{1\le k\le n} \mathbb{E}(Y_{k-1}) + 2n \\
-)\qquad n\,\mathbb{E}(Y_{n-1}) &= \sum_{1\le k\le n-1} \mathbb{E}(Y_{k-1}) + 2(n-1)
\end{aligned}
$$

$$(n+1)\,\mathbb{E}(Y_n) = (n+1)\,\mathbb{E}(Y_{n-1}) + 2$$

$$\Rightarrow \begin{cases} \mathbb{E}(Y_n) & = \mathbb{E}(Y_{n-1}) + \frac{2}{n+1} \\ \mathbb{E}(Y_{n-1}) & = \mathbb{E}(Y_{n-2}) + \frac{2}{n} \\ & \vdots \\ \mathbb{E}(Y_2) & = \mathbb{E}(Y_1) + \frac{2}{3} \\ \mathbb{E}(Y_1) & = \frac{2}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \mathbb{E}(Y_n) & = 2H_{n+1} - 2 \\ \mathbb{E}(X_n) & = \frac{n+1}{n} E(Y_n) - 1 = 2\frac{n+1}{n} H_n - 3 \end{cases} \qquad \blacksquare$$

Lemma 2. $\mathbb{E}(X_n) = 1 + \frac{1}{n} \sum_{1 \le k \le n} \mathbb{E}(Y_{k-1})$

Proof:

$$\begin{aligned} \mathbb{E}(X_n) &= \frac{1}{n!\,n} \sum_{(x_1,\ldots,x_n)} \sum_{1 \le k \le n} X_n(x_1, \ldots, x_n; k) \\ &= \frac{1}{n!\,n} \sum_{(x_1,\ldots,x_n) \in S_n} \sum_{1 \le k \le n} \Big(1 + Y_n(x_1, \ldots, x_{k-1}; k)\Big) \\ &= 1 + \frac{1}{n!\,n} \sum_{1 \le k \le n} \sum_{(x_1,\ldots,x_k) \in S_k} n \cdots (k+1) Y_{k-1}(x_1, \ldots, x_{k-1}; x_k) \\ &= 1 + \frac{1}{n} \sum_{1 \le k \le n} \left[\frac{1}{k!} \sum_{(x_1,\ldots,x_k) \in S_k} Y_{k-1}(x_1, \ldots, x_{k-1}; x_k)\right] \\ &= 1 + \frac{1}{n} \sum_{1 \le k \le n} \mathbb{E}(Y_{k-1}) \qquad \blacksquare \end{aligned}$$