Use **guessing method** to find the *leading term* from recurrence:

Example 1. $P(n) = n! \quad \Rightarrow \quad \text{(Recurrence) } P(n) = nP(n-1)$

Guess:

$P(n) = n^c$? $\quad n^c \stackrel{?}{=} n(n-1)^c \stackrel{?}{=} n^{c+1} + \mathcal{O}(n^c)$ — LHS is smaller; n^c is too small

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$P(n) = a^n$? $\quad a^n \stackrel{?}{=} na^{n-1} \stackrel{?}{=} \frac{n}{a}a^n$ — LHS is smaller; a^n is too small

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$P(n) = n^n$? $\quad n^n \stackrel{?}{=} n(n-1)^{n-1} = n^n(1-\frac{1}{n})^{n-1} \stackrel{?}{=} n^n e^{-1}$ — LHS is bigger; n^n is too big

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$$P(n) = \left(\frac{n}{a}\right)^{n+c}, \quad \left(\frac{n}{a}\right)^{n+c} \stackrel{?}{=} n\left(\frac{n-1}{a}\right)^{n-1+c} = \left(\frac{n}{a}\right)^{n+c} \cdot a\left(1-\frac{1}{n}\right)^{n-1+c}$$

$$\begin{aligned}
\left(1-\frac{1}{n}\right)^{n-1+c} &= e^{(n-1+c)\ln\left(1-\frac{1}{n}\right)} \\
&= e^{(n-1+c)\left(-\frac{1}{n}-\frac{1}{2n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right)} \\
&= e^{-1+(1-c-\frac{1}{2})\frac{1}{n}+\mathcal{O}(\frac{1}{n^2})} \\
&= e^{-1}e^{(\frac{1}{2}-c)\frac{1}{n}+\mathcal{O}(\frac{1}{n^2})} \\
&= e^{-1}\left[1+\left(\frac{1}{2}-c\right)\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right]
\end{aligned}$$

$$\therefore \begin{cases} a = e \\ c = \frac{1}{2} \end{cases} \implies P(n) = \Theta\left(\left(\frac{n}{e}\right)^{n+\frac{1}{2}}\right)$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\cdots\right)$$

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Appendix (as $x \to \infty$)

- $(1+\frac{1}{x})^x \to e^x$
- $(1+\frac{c}{x})^x = (1+\frac{1}{x/c})^{\frac{x}{c}\cdot c} \to e^c$

Example 2. $\square_n = 1^2 + 2^2 + \cdots + n^2 \quad \Rightarrow \quad \square_n = \square_{n-1} + n^2$

Guess:

$$\square_n = a \cdot n^c \; ? \qquad an^c \stackrel{?}{=} a(n-1)^c + n^2$$
$$\stackrel{?}{=} an^c - acn^{c-1} + n^2 + \mathcal{O}(n^{c-2})$$

$$\therefore \begin{cases} c = 3 \\ a = \frac{1}{3} \end{cases} \implies \square_n = \frac{1}{3}n^3 + \mathcal{O}(n^2) \qquad \blacksquare$$

Example 3. $T(n) = T\left(\frac{1}{4}n\right) + T\left(\frac{3}{4}n\right) + n$

Guess:

$T(n) = n$?

$$n \stackrel{?}{=} \frac{1}{4}n + \frac{3}{4}n + n \qquad \text{LHS is smaller}$$
$$\stackrel{?}{=} 2n \qquad n \text{ is too small}$$

$T(n) = n^2$?

$$n^2 \stackrel{?}{=} \left(\frac{1}{4}n\right)^2 + \left(\frac{3}{4}n\right)^2 + n \qquad \text{LHS is bigger}$$
$$\stackrel{?}{=} \frac{10}{16}n^2 + n \qquad n^2 \text{ is too big}$$

$T(n) = an\log n$?

$$an\log n \stackrel{?}{=} a\frac{n}{4}\log\left(\frac{n}{4}\right) + a\frac{3n}{4}\log\left(\frac{3n}{4}\right) + n$$
$$\stackrel{?}{=} a\frac{n}{4}(\log n - \log 4) + a\frac{3n}{4}(\log n + \log 3 - \log 4) + n$$
$$\stackrel{?}{=} an\log n + a\frac{n}{4}\Big[-\log 4 + 3(\log 3 - \log 4)\Big] + n$$
$$\stackrel{?}{=} an\log n + \frac{n}{4}\Big[a(3\log 3 - 4\log 4) + 4\Big]$$

$$\Rightarrow a(3\log 3 - 4\log 4) + 4 = 0$$

$$\therefore \; T(n) = an\log n, \quad \text{where} \quad a = 4/(\log 256 - \log 27) \qquad \blacksquare$$