Z-Transform

- Fourier Transform

\[ X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} \]

- z-transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]
Z-Transform (continue)

- Z-transform operator: \( Z\{ \cdot \} \)
  \[
  Z\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k] z^{-k} = X(z)
  \]

- The z-transform operator is seen to transform the sequence \( x[n] \) into the function \( X\{z\} \), where \( z \) is a continuous complex variable.
  - From time domain (or space domain, n-domain) to the z-domain
    \[
    x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \leftrightarrow Z\{x[n]\} = X(Z)
    \]
Bilateral vs. Unilateral

- Two sided or bilateral z-transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

- Unilateral z-transform

\[ X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \]
Example of z-transform

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n \leq -1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$N &gt; 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \]
If we replace the complex variable $z$ in the $z$-transform by $e^{jw}$, then the $z$-transform reduces to the Fourier transform.

The Fourier transform is simply the $z$-transform when evaluating $X(z)$ in a unit circle in the $z$-plane.

Generally, we can express the complex variable $z$ in the polar form as $z = re^{jw}$. With $z$ expressed in this form,

$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n](re^{jw})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-jwn}$$
In this sense, the z-transform can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence $r^{-n}$.

For $r=1$, the z-transform reduces to the Fourier transform.
Relationship to the Fourier Transform (continue)

- Beginning at \( z = 1 \) (i.e., \( w = 0 \)) through \( z = j \) (i.e., \( w = \pi/2 \)) to \( z = -1 \) (i.e., \( w = \pi \)), we obtain the Fourier transform from \( 0 \leq w \leq \pi \).

- Continuing around the unit circle in the \( z \)-plane corresponds to examining the Fourier transform from \( w = \pi \) to \( w = 2\pi \).
  - Fourier transform is usually displayed on a linear frequency axis. Interpreting the Fourier transform as the \( z \)-transform on the unit circle in the \( z \)-plane corresponds conceptually to wrapping the linear frequency axis around the unit circle.
Convergence Region of Z-transform

- Region of convergence (ROC)

  - Since the z-transform can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence $r^{-n}$, it is possible for the z-transform to converge even if the Fourier transform does not.

  - Because

    $$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \leq \sum_{n=-\infty}^{\infty} |x[n]|z^{-n}$$

  - $X(z)$ is convergent (i.e. bounded) i.e., $\Sigma x[n]r^{-n} < \infty$, if $x[n]$ is absolutely summable.

  - Eg., $x[n] = u[n]$ is absolutely summable if $r>1$. This means that the z-transform for the unit step exists with ROC $|z|>1$. 
ROC of Z-transform

- In fact, convergence of the power series $X(z)$ depends only on $|z|$.

$$\sum_{n=-\infty}^{\infty} |x[n]| z^{-n} < \infty$$

- If some value of $z$, say $z = z_1$, is in the ROC, then all values of $z$ on the circle defined by $|z| = |z_1|$ will also be in the ROC.

- Thus the ROC will consist of a ring in the $z$-plane.
ROC of Z-transform – Ring Shape
Analytic Function and ROC

- The $z$-transform is a Laurent series of $z$.
  - A number of elegant and powerful theorems from the complex-variable theory can be employed to study the $z$-transform.
  - A Laurent series, and therefore the $z$-transform, represents an *analytic* function at every point inside the region of convergence.
  - Hence, the $z$-transform and all its derivatives exist and must be continuous functions of $z$ with the ROC.
  - This implies that if the ROC includes the unit circle, the Fourier transform and all its derivatives with respect to $w$ must be continuous function of $w$. 
Z-transform and Linear Systems

- Z-transform of a causal FIR system

\[ y[n] = \sum_{m=0}^{M} b_m x[n-m] \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n&lt;0 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>( M )</th>
<th>( N&gt;M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h[n] )</td>
<td>0</td>
<td>( b_0 )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>...</td>
<td>( b_M )</td>
<td>0</td>
</tr>
</tbody>
</table>

- The impulse response is

\[ h[n] = \sum_{m=0}^{M} b_m \delta[n-m] \]

- Take the z-transform on both sides
Z-transform of Causal FIR System (continue)

\[ Y(z) = Z\{y[n]\} = Z\left\{ \sum_{m=0}^{M} b_m x[n-m] \right\} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M} b_m x[n-m] z^{-n} \]

\[ = \sum_{m=0}^{M} b_m \sum_{n=-\infty}^{\infty} x[n-m] z^{-n} = \sum_{m=0}^{M} b_m z^{-m} \left( \sum_{n=-\infty}^{\infty} x[n-m] z^{-(n-m)} \right) \]

\[ = \sum_{m=0}^{M} b_m z^{-m} Z\{x[n]\} = Z\{h[n]\} X[z] = H(z)X(z) \]

Thus, the z-transform of the output of a FIR system is the product of the z-transform of the input signal and the z-transform of the impulse response.
Z-transform of Causal FIR System (continue)

\[ H(z) = \sum_{m=0}^{M} b_m z^{-m} \]

- \( H(z) \) is called the *system function* (or transfer function) of a (FIR) LTI system.

![Diagram](image-url)
Multiplication Rule of Cascading System

\[ X(z) \xrightarrow{H_1(z)} Y(z) \xrightarrow{H_2(z)} V(z) \]

\[ \equiv \]

\[ X(z) \xrightarrow{H_1(z)} Y(z) \xrightarrow{H_2(z)} V(z) \]

\[ \equiv \]

\[ X(z) \xrightarrow{H_1(z)H_2(z)} Y(z) \]
Example

- Consider the FIR system \( y[n] = 6x[n] - 5x[n-1] + x[n-2] \)
- The z-transform system function is

\[
H(z) = 6 - 5z^{-1} + z^{-2}
\]

\[
= (3 - z^{-1})(2 - z^{-1}) = 6 \left( \frac{z - \frac{1}{3}}{z} \right) \left( \frac{z - \frac{1}{2}}{z} \right)
\]
Delay of one Sample

- Consider the FIR system $y[n] = x[n-1]$, i.e., the one-sample-delay system.
- The z-transform system function is

$$H(z) = z^{-1}$$
Delay of k Samples

- Similarly, the FIR system $y[n] = x[n - k]$, i.e., the k-sample-delay system, is the z-transform of the impulse response $\delta[n - k]$.

$$H(z) = z^{-k}$$

\[ z^{-k} \]
The signal-flow graph of a causal FIR system can be re-represented by z-transforms.
Z-transform of General Difference Equation

- Remember that the general form of a linear constant-coefficient difference equation is

\[
\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \quad \text{for all } n
\]

- When \( a_0 \) is normalized to \( a_0 = 1 \), the system diagram can be shown as below
Review of Linear Constant-coefficient Difference Equation

\[ x[n] \rightarrow b_0 \rightarrow \ldots \rightarrow b_M \rightarrow \ldots \rightarrow y[n] \]

\[ x[n-1] \rightarrow b_1 \rightarrow \ldots \rightarrow b_M \rightarrow \ldots \rightarrow y[n-1] \]

\[ x[n-2] \rightarrow b_2 \rightarrow \ldots \rightarrow b_M \rightarrow \ldots \rightarrow y[n-2] \]

\[ x[n-M] \rightarrow b_M \rightarrow \ldots \rightarrow b_M \rightarrow \ldots \rightarrow y[n-M] \]
Z-transform of Linear Constant-coefficient Difference Equation

The signal-flow graph of difference equations represented by z-transforms.

\[
X(z) \quad b_0 \quad + \quad + \quad + \quad Y(z)
\]

\[
z^{-1} \quad b_1 \quad + \quad + \quad -a_1 \quad z^{-1}
\]

\[
z^{-1} \quad b_2 \quad + \quad + \quad -a_2 \quad z^{-1}
\]

\[
z^{-1} \quad b_M \quad + \quad + \quad -a_N \quad z^{-1}
\]
Z-transform of Difference Equation (continue)

- From the signal-flow graph,
  \[ Y(z) = \sum_{m=0}^{M} b_m X(z) z^{-m} - \sum_{k=1}^{N} a_k Y(z) z^{-k} \]

- Thus,
  \[ \sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{m=0}^{M} b_m X(z) z^{-m} \]

- We have
  \[ \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}} \]
Let \( H(z) \) is called the system function of the LTI system defined by the linear constant-coefficient difference equation.

The multiplication rule still holds: \( Y(z) = H(z)X(z) \), i.e.,
\[
Z\{y[n]\} = H(z)Z\{x[n]\}.
\]

The system function of a difference equation is a rational form \( X(z) = P(z)/Q(z) \).

Since LTI systems are often realized by difference equations, the rational form is the most common and useful for z-transforms.
When $a_k = 0$ for $k = 1 \ldots N$, the difference equation degenerates to a FIR system we have investigated before.

$$H(z) = \sum_{m=0}^{M} b_m z^{-m}$$

It can still be represented by a rational form of the variable $z$ as

$$H(z) = \sum_{m=0}^{M} b_m z^{-M-m}$$

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{M-m}}{z^M}$$
When the input \( x[n] = \delta[n] \), the z-transform of the impulse response satisfies the following equation:

\[
Z\{h[n]\} = H(z)Z\{\delta[n]\}.
\]

Since the z-transform of the unit impulse \( \delta[n] \) is equal to one, we have

\[
Z\{h[n]\} = H(z)
\]

That is, the system function \( H(z) \) is the z-transform of the impulse response \( h[n] \).
System Function and Impulse Response (continue)

Generally, for a linear system,

\[ y[n] = T\{x[n]\} \]

it can be shown that

\[ Y\{z\} = H(z)X(z). \]

where \( H(z) \), the system function, is the \( z \)-transform of the impulse response of this system \( T\{\cdot\} \).

Also, cascading of systems becomes multiplication of system function under \( z \)-transforms.

\[
\begin{array}{ccc}
X(z) & \xrightarrow{H(z)/H(e^{jw})} & Y(z) (= H(z)X(z)) \\
X(e^{jw}) & \xrightarrow{H(z)/H(e^{jw})} & Y(e^{jw}) (= H(e^{jw})X(e^{jw})) \\
\end{array}
\]

Z-transform

Fourier transform
Poles and Zeros

- **Pole:**
  - The *pole* of a z-transform $X(z)$ are the values of $z$ for which $X(z) = \infty$.

- **Zero:**
  - The *zero* of a z-transform $X(z)$ are the values of $z$ for which $X(z) = 0$.

- When $X(z) = P(z)/Q(z)$ is a rational form, and both $P(z)$ and $Q(z)$ are polynomials of $z$, the poles of $X(z)$ are the roots of $Q(z)$, and the zeros are the roots of $P(z)$, respectively.
Zeros of a system function

The system function of the FIR system $y[n] = 6x[n] - 5x[n-1] + x[n-2]$ has been shown as

$$H(z) = 6 \frac{(z - \frac{1}{3})(z - \frac{1}{2})}{z^2} = \frac{P(z)}{Q(z)}$$

The zeros of this system are $1/3$ and $1/2$, and the pole is 0.

Since 0 and 0 are double roots of $Q(z)$, the pole is a second-order pole.
Example: Right-sided Exponential Sequence

- **Right-sided sequence:**
  - A discrete-time signal is right-sided if it is nonzero only for \( n \geq 0 \).

- Consider the signal \( x[n] = a^n u[n] \).

\[
X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n
\]

- For convergent \( X(z) \), we need \( \sum_{n=0}^{\infty} (az^{-1})^n < \infty \)

  - Thus, the ROC is the range of values of \( z \) for which \( |az^{-1}| < 1 \) or, equivalently, \( |z| > a \).
Example: Right-sided Exponential Sequence (continue)

- By sum of power series,
  \[
  X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|
  \]

- There is one zero, at \(z=0\), and one pole, at \(z=a\).

\(\circ\) : zeros  
\(\times\) : poles

Gray region: ROC
Example: Left-sided Exponential Sequence

- Left-sided sequence:
  - A discrete-time signal is left-sided if it is nonzero only for $n \leq -1$.

- Consider the signal $x[n] = -a^n u[-n-1]$.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

- If $|az^{-1}| < 1$ or, equivalently, $|z| < a$, the sum converges.
Example: Left-sided Exponential Sequence (continue)

- By sum of power series,
  \[ X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{z}{z - a}, \quad |z| < |a| \]

- There is one zero, at \( z=0 \), and one pole, at \( z=a \).

The pole-zero plot and the algebraic expression of the system function are the same as those in the previous example, but the ROC is different.
Example: Sum of Two Exponential Sequences

Given

\[ x(n) = \left( \frac{1}{2} \right)^n u(n) + \left( -\frac{1}{3} \right)^n u(n) \]

Then

\[ X(z) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2} \right)^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} \left( -\frac{1}{3} \right)^n u(n) z^{-n} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} + \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n z^{-n} \]

\[ = \frac{1}{1-z^{-1}/2} + \frac{1}{1+z^{-1}/3} = \frac{2z \left( z - \frac{1}{12} \right)}{\left( z - \frac{1}{2} \right) \left( z + \frac{1}{3} \right)} \]
Example: Sum of Two Exponential Sequences (continue)

\[
\left(\frac{1}{2}\right)^n u(n) \leftrightarrow z \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}
\]

\[
\left(-\frac{1}{3}\right)^n u(n) \leftrightarrow z \frac{2}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}
\]

Thus

\[
\left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n) \leftrightarrow z \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{2}
\]
Example: Sum of Two Exponential Sequences (continue)
Example: Two-sided Exponential Sequence

Given \( x(n) = \left( -\frac{1}{3} \right)^n u(n) - \left( \frac{1}{2} \right)^n u(-n-1) \)

Since \( \left( -\frac{1}{3} \right)^n u(n) \leftrightarrow \frac{1}{1 + \frac{1}{3} z^{-1}}, \quad |z| > \frac{1}{3} \)

and by the left-sided sequence example

\( -\left( \frac{1}{2} \right)^n u(-n-1) \leftrightarrow \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| < \frac{1}{2} \)
Example: Two-sided Exponential Sequence (continue)

\[ X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z + \frac{1}{3})(z - \frac{1}{2})} \]

Again, the poles and zeros are the same as the previous example, but the ROC is not.
Example: Finite-length Sequence
(FIR System)

Given
\[ x(n) = \begin{cases} 
  a^n & 0 \leq n \leq N - 1 \\
  0 & \text{otherwise} 
\end{cases} \]

Then
\[ X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1 - \left(az^{-1}\right)^N}{1 - az^{-1}} \]

\[ = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a} \]

There are the \( N \) roots of \( z^N = a^N, z_k = ae^{j(2\pi k/N)}. \) The root of \( k = 0 \) cancels the pole at \( z=a. \) Thus there are \( N-1 \) zeros, \( z_k = ae^{j(2\pi k/N)}, k = 1 \ldots N, \) and a \((N-1)\)th order pole at zero.
Pole-zero Plot

15th-order pole

Unit circle

$\pi/8$
Some Common Z-transform Pairs

$$\delta[n] \leftrightarrow 1$$  ROC: all $z$.

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}$$  ROC: $|z| > 1$.

$$-u[-n-1] \leftrightarrow \frac{1}{1 - z^{-1}}$$  ROC: $|z| < 1$.

$$\delta[n-m] \leftrightarrow z^{-m}$$  ROC: all $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$).

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}$$  ROC: $|z| > |a|$.

$$-a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}$$  ROC: $|z| < |a|$.

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1 - az^{-1})^2}$$  ROC: $|z| > |a|$. 
Some Common Z-transform Pairs (continue)

\(-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}\)

ROC: \(|z| < |a|.

\([\cos w_0 n] u[n] \leftrightarrow \frac{1-[\cos w_0]z^{-1}}{1-[2\cos w_0]z^{-1} + z^{-2}}\)

ROC: \(|z| > 1.\)

\([\sin w_0 n] u[n] \leftrightarrow \frac{[\sin w_0]z^{-1}}{1-[2\cos w_0]z^{-1} + z^{-2}}\)

ROC: \(|z| > 1.\)

\([r^n \cos w_0 n] u[n] \leftrightarrow \frac{1-[r \cos w_0]z^{-1}}{1-[2r \cos w_0]z^{-1} + r^2 z^{-2}}\)

ROC: \(|z| > r.\)

\([r^n \sin w_0 n] u[n] \leftrightarrow \frac{[r \sin w_0]z^{-1}}{1-[2r \cos w_0]z^{-1} + r^2 z^{-2}}\)

ROC: \(|z| > r.\)

\(\begin{cases} 
a^n & 0 \leq n \leq N-1 \\
0 & \text{otherwise}
\end{cases} \leftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}}\)

ROC: \(|z| > 0.\)
Properties of the ROC

- The ROC is a ring or disk in the z-plane centered at the origin; i.e., \( 0 \leq r_R < |z| \leq r_L \leq \infty \).
- The Fourier transform of \( x[n] \) converges absolutely iff the ROC includes the unit circle.
- The ROC cannot contain any poles.
- If \( x[n] \) is a finite duration sequence, then the ROC is the entire z-plane except possible \( z=0 \) or \( z=\infty \).
- If \( x[n] \) is a right-sided sequence, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in \( X(z) \) to (and possibly include) \( z=\infty \).
Properties of the ROC (continue)

- If $x[n]$ is a left-sided sequence, the ROC extends inward from the innermost (i.e., smallest magnitude) nonzero pole in $X(z)$ to (and possibly include) $z = 0$.

- A two-sided sequence $x[n]$ is an infinite-duration sequence that is neither right nor left sided. The ROC will consist of a ring in the $z$-plane, bounded on the interior and exterior by a pole, but not containing any poles.

- The ROC must be a connected region.
Properties of the ROC (continue)

Consider the system function $H(z)$ of a linear system:

- If the system is stable, the impulse response $h(n)$ is absolutely summable and therefore has a Fourier transform, then the ROC must include the unit circle.

- If the system is causal, then the impulse response $h(n)$ is right-sided, and thus the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $H(z)$ to (and possibly include) $z = \infty$. 
**Inverse Z-transform**

- Given $X(z)$, find the sequence $x[n]$ that has $X(z)$ as its z-transform.
- We need to specify both algebraic expression and ROC to make the inverse Z-transform unique.
- Techniques for finding the inverse z-transform:
  - **Investigation method:**
    - By inspect certain transform pairs.
    - Eg. If we need to find the inverse z-transform of
      \[
      X(z) = \frac{1}{1 - 0.5z^{-1}}
      \]
      From the transform pair we see that $x[n] = 0.5^n u[n]$. 

Inverse Z-transform by Partial Fraction Expansion

- If $X(z)$ is the rational form with

$$X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- An equivalent expression is

$$X(z) = \frac{z^{-M} \sum_{m=0}^{M} b_m z^{M-m}}{z^{-N} \sum_{k=0}^{N} a_k z^{N-k}} = \frac{z^{N} \sum_{m=0}^{M} b_m z^{M-m}}{z^{M} \sum_{k=0}^{N} a_k z^{N-k}}$$
Inverse Z-transform by Partial Fraction Expansion (continue)

- There will be $M$ zeros and $N$ poles at nonzero locations in the $z$-plane.
- Note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \prod_{m=1}^{M} \left(1 - c_m z^{-1}\right) \prod_{m=1}^{N} \left(1 - d_k z^{-1}\right)$$

where $c_k$’s and $d_k$’s are the nonzero zeros and poles, respectively.
Then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Obviously, the common denominators of the fractions in the above two equations are the same. Multiplying both sides of the above equation by $1 - d_k z^{-1}$ and evaluating for $z = d_k$ shows that

$$A_k = \left(1 - d_k z^{-1}\right)X(z)\bigg|_{z=d_k}$$
Example

- Find the inverse z-transform of

$$X(z) = \frac{1}{(1 - (1/4)z^{-1})(1 - (1/2)z^{-1})} \quad |z| > \frac{1}{2}$$

$X(z)$ can be decomposed as

$$X(z) = \frac{A_1}{(1 - (1/4)z^{-1})} + \frac{A_2}{(1 - (1/2)z^{-1})}$$

Then

$$A_1 = \left(1 - (1/4)z^{-1}\right)X(z)\bigg|_{z=1/4} = -1$$

$$A_2 = \left(1 - (1/2)z^{-1}\right)X(z)\bigg|_{z=1/2} = 2$$
Thus

\[ X(z) = \frac{-1}{(1-(1/4)z^{-1})} + \frac{2}{(1-(1/2)z^{-1})} \]

and so

\[ x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n] \]
Another Example

Find the inverse z-transform of

\[ X(z) = \frac{(1+z^{-1})^2}{\left(1-(1/2)z^{-1}\right)(1-z^{-1})} \quad |z| > 1 \]

Since both the numerator and denominator are of degree 2, a constant term exists.

\[ X(z) = B_0 + \frac{A_1}{1-(1/2)z^{-1}} + \frac{A_2}{1-z^{-1}} \]

\( B_0 \) can be determined by the fraction of the coefficients of \( z^{-2} \), \( B_0 = 1/(1/2) = 2 \).
Another Example (continue)

\[ X(z) = 2 + \frac{A_1}{\left(1 - \left(1/2\right)z^{-1}\right)} + \frac{A_2}{\left(1 - z^{-1}\right)} \]

\[ A_1 = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \left(1/2\right)z^{-1}\right)\left(1 - z^{-1}\right)} \left(1 - \left(1/2\right)z^{-1}\right) \bigg|_{z = 1/2} = 9 \]

\[ A_2 = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \left(1/2\right)z^{-1}\right)\left(1 - z^{-1}\right)} \left(1 - z^{-1}\right) \bigg|_{z = 1} = 8 \]

Therefore

\[ X(z) = 2 - \frac{9}{\left(1 - \left(1/2\right)z^{-1}\right)} + \frac{8}{\left(1 - z^{-1}\right)} \]

\[ x[n] = 2\delta[n] - 9\left(1/2\right)^n u[n] + 8u[n] \]
We can determine any particular value of the sequence by finding the coefficient of the appropriate power of $z^{-1}$. 

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

\[ = ... + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + ... \]
Example: Finite-length Sequence

Find the inverse $z$-transform of

$$X(z) = z^2 \left(1 - 0.5z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

By directly expand $X(z)$, we have

$$X(z) = z^2 - 0.5z - 1 + 0.5z^{-1}$$

Thus,

$$x[n] = \delta[n + 2] - 0.5\delta[n + 1] - \delta[n] + 0.5\delta[n - 1]$$
Example

Find the inverse z-transform of

\[ X(z) = \log(1 + az^{-1}) \quad |z| > |a| \]

Using the power series expansion for \( \log(1+x) \) with \(|x|<1\), we obtain

\[ X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \]

Thus

\[ x[n] = \begin{cases} 
(-1)^{n+1} a^n / n & n \geq 1 \\
0 & n \leq 0 
\end{cases} \]
Z-transform Properties

- **Suppose**

\[
x[n] \leftrightarrow X(z) \quad \text{ROC} = R_x
\]

\[
x_1[n] \leftrightarrow X_1(z) \quad \text{ROC} = R_{x_1}
\]

\[
x_2[n] \leftrightarrow X_2(z) \quad \text{ROC} = R_{x_2}
\]

- **Linearity**

\[
a x_1[n] + b x_2[n] \leftrightarrow aX_1(z) + bX_2(z) \quad \text{ROC} = R_{x_1} \cap R_{x_2}
\]
Time shifting

\[ x[n - n_0] \xrightarrow{z} z^{-n_0} X(z) \quad \text{ROC} = R_x \] (except for the possible addition or deletion of \( z = 0 \) or \( z = \infty \).

Multiplication by an exponential sequence

\[ z_0^n x[n] \xrightarrow{z} X(z / z_0) \quad \text{ROC} = |z_0| R_x \]
Z-transform Properties (continue)

- Differentiation of $X(z)$
  \[ nx[n] \iff -z \frac{dX(z)}{dz} \quad \text{ROC} = R_x \]

- Conjugation of a complex sequence
  \[ x^*[n] \iff X^*(z^*) \quad \text{ROC} = R_x \]
Z-transform Properties (continue)

- **Time reversal**

\[ x^*[-n] \leftrightarrow X^*(1/z^*) \quad \text{ROC} = \frac{1}{R_x} \]

If the sequence is real, the result becomes

\[ x[-n] \leftrightarrow X(1/z) \quad \text{ROC} = \frac{1}{R_x} \]

- **Convolution**

\[ x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \quad \text{ROC contains } R_{x_1} \cap R_{x_2} \]
Z-transform Properties
(continue)

- Initial-value theorem: If \( x[n] \) is zero for \( n<0 \) (i.e., if \( x[n] \) is causal), then

\[
x[0] = \lim_{z \to \infty} X(z)
\]