In summary

If \( x[n] \) is a finite-length sequence (\( n \neq 0 \) only when \( |n| < N \)), its DTFT \( X(e^{jw}) \) shall be a periodic continuous function with period \( 2\pi \).

The DFT of \( x[n] \), denoted by \( X(k) \), is also of length \( N \).

\[
x(n) = \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn} = \sum_{k=0}^{N-1} X(k) W^{-kn}, \quad 0 \leq n \leq N - 1
\]

\[
X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) W^{kn}, \quad 0 \leq k \leq N - 1
\]

where \( W = e^{-j(2\pi/N)n} \), and \( W^n \) are the roots of \( W^n = 1 \).

- Relationship: \( X(k) \) is the uniform samples of \( X(e^{jw}) \) at the discrete frequency \( w_k = (2\pi/N)k \), when the frequency range \([0, 2\pi]\) is divided into \( N \) equally spaced points.
The Concept of ‘System’
(oppenheim et al. 1999)

• Discrete-time Systems
  – A transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with value $y[n]$.

$$y[n] = T\{x[n]\}$$

\[
x[n] \rightarrow \boxed{T\{\cdot\}} \rightarrow y[n]
\]
System Examples

• Ideal Delay
  - \( y[n] = x[n-n_d] \), where \( n_d \) is a fixed positive integer called the delay of the system.

• Moving Average
  \[
  y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]
  \]

• Memoryless Systems
  - The output \( y[n] \) at every value of \( n \) depends only on the input \( x[n] \), at the same value of \( n \).
  - Eg. \( y[n] = (x[n])^2 \), for each value of \( n \).
System Examples (continue)

• Linear System: If \( y_1[n] \) and \( y_2[n] \) are the responses of a system when \( x_1[n] \) and \( x_2[n] \) are the respective inputs. The system is linear if and only if
  - \( T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \).
  - \( T\{ax[n]\} = aT\{x[n]\} = ay[n] \), for arbitrary constant \( a \).
  - So, if \( x[n] = \sum_k a_k x_k[n] \), \( y[n] = \sum_k a_k y_k[n] \) (superposition principle)

• For example

Accumulator System

\[
y[n] = \sum_{k=\infty}^{n} x[k] \quad \text{(is a linear system)}
\]
System Examples (continue)

• Nonlinear System.
  – Eg. $w[n] = \log_{10}(|x[n]|)$ is not linear.

• Time-invariant System:
  – If $y[n] = T\{x[n]\}$, then $y[n-n_0] = T\{x[n-n_0]\}$
  – The accumulator is a time-invariant system.

• The compressor system (not time-invariant)
  – $y[n] = x[Mn], -\infty < n < \infty$. 
System Examples (continue)

• Causality
  – A system is causal if, for every choice of $n_0$, the output sequence value at the index $n = n_0$ depends only the input sequence values for $n \leq n_0$.
  – That is, if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$.

• Eg. Forward-difference system (non causal)
  – $y[n] = x[n+1] - x[n]$ (The current value of the output depends on a future value of the input)

• Eg. Background-difference (causal)
  – $y[n] = x[n] - x[n-1]$
System Examples (continue)

• Stability
  – Bounded input, bounded output (BIBO): If the input is bounded, \(|x[n]| \leq B_x < \infty\) for all \(n\), then the output is also bounded, i.e., there exists a positive value \(B_y\) s.t. \(|y[n]| \leq B_y < \infty\) for all \(n\).

• Eg., the system \(y[n] = (x[n])^2\) is stable.

• Eg., the accumulated system is unstable, which can be easily verified by setting \(x[n] = u[n]\), the unit step signal.
Linear Time Invariant Systems

• A system that is both linear and time invariant is called a linear time invariant (LTI) system.

• By setting the input $x[n]$ as $\delta[n]$, the impulse function, the output $h[n]$ of an LTI system is called the impulse response of this system.
  
  – Time invariant: when the input is $\delta[n-k]$, the output is $h[n-k]$.
  
  – Remember that the $x[n]$ can be represented as a linear combination of delayed impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$$
Linear Time Invariant Systems (continue)

• Hence

\[ y[n] = T \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \right\} = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\} \]

\[ = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]

• Therefore, a LTI system is completely characterized by its impulse response \( h[n] \).
Linear Time Invariant Systems (continue)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

- Note that the above operation is convolution, and can be written in short by $y[n] = x[n] \ast h[n]$.
- The output of an LTI system is equivalent to the convolution of the input and the impulse response.

- In a LTI system, the input sample at $n = k$, represented as $x[k] \delta[n-k]$, is transformed by the system into an output sequence $x[k]h[n-k]$ for $-\infty < n < \infty$. 
Property of LTI System and Convolution

• Communitive
  \[ x[n] * h[n] = h[n] * x[n]. \]

• Distributive over addition
  \[ x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]. \]

• Cascade connection

\[
\begin{align*}
&\rightarrow h_1[n] \quad \rightarrow h_2[n] \quad \rightarrow y[n] \\
&\rightarrow h_2[n] \quad \rightarrow h_1[n] \quad \rightarrow y[n] \\
&\rightarrow x[n] \quad \rightarrow h_1[n] \ast h_2[n] \quad \rightarrow y[n]
\end{align*}
\]
Property of LTI System and Convolution (continue)

- Parallel combination of LTI systems and its equivalent system.
Property of LTI System and Convolution (continue)

- Stability: A LTI system is stable if and only if

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Since

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]| < \infty$$

when $|x[n]| \leq B_x$.

- This is a sufficient condition proof.
Property of LTI System and Convolution (continue)

- **Causality**
  - those systems for which the output depends only on the input samples \(y[n_0]\) depends only the input sequence values for \(n \leq n_0\).
  - Follow this property, an LTI system is causal iff
    \[ h[n] = 0 \quad \text{for all } n < 0. \]
  - Causal sequence: a sequence that is zero for \(n<0\). A causal sequence could be the impulse response of a causal system.
Impulse Responses of Some LTI Systems

- Ideal delay: \( h[n] = \delta[n-n_d] \)
- Moving average
  \[
  h[n] = \begin{cases} 
  \frac{1}{M_1 + M_2 + 1} & -M_1 \leq n \leq M_2 \\
  0 & \text{otherwise}
  \end{cases}
  \]
- Accumulator
  \[
  h[n] = \begin{cases} 
  1 & n \geq 0 \\
  0 & \text{otherwise}
  \end{cases}
  \]
- Forward difference: \( h[n] = \delta[n+1] - \delta[n] \)
- Backward difference: \( h[n] = \delta[n] - \delta[n-1] \)
Examples of Stable/Unstable Systems

- In the above, moving average, forward difference and backward difference are stable systems, since the impulse response has only a finite number of terms.
  - Such systems are called finite-duration impulse response (FIR) systems.
  - FIR is equivalent to a weighted average of a sliding window.
  - FIR systems will always be stable.
- The accumulator is unstable since $S = \sum_{n=0}^{\infty} u[n] = \infty$
Examples of Stable/Unstable Systems (continue)

• When the impulse response is infinite in duration, the system is referred to as an infinite-duration impulse response (IIR) system.
  - The accumulator is an IIR system.

• Another example of IIR system: \( h[n] = a^n u[n] \)
  - When \( |a| < 1 \), this system is stable since
    \[
    S = 1 + |a| + |a|^2 + \ldots + |a|^n + \ldots = 1/(1-|a|) \text{ is bounded.}
    \]
  - When \( |a| \geq 1 \), this system is unstable
Examples of Causal Systems

- The ideal delay, accumulator, and backward difference systems are causal.
- The forward difference system is noncausal.
- The moving average system is causal requires $-M_1 \geq 0$ and $M_2 \geq 0$. 
A LTI system can be realized in different ways by separating it into different subsystems.

\[
h[n] = (\delta[n+1] - \delta[n]) * \delta[n-1] \\
= \delta[n-1] * (\delta[n+1] - \delta[n]) \\
= \delta[n] - \delta[n-1]
\]
Equivalent Systems (continue)

- Another example of cascade systems – inverse system.

\[ h[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n] \]
Linear Constant-coefficient Difference Equations

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \quad \text{for all } n \]

- An important subclass of LTI systems consist of those system for which the input \( x[n] \) and output \( y[n] \) satisfy an \( N \)th-order linear constant-coefficient difference equation.

- A general form is shown above.

- Not-all LTI systems can be represented into this form, but it specifies a wide class of LTI systems.
Block Diagram of the Difference Equation

- Assume that $a_0 = 1$. Let TD denote *one-sample delay*.
Difference Equation: FIR system

- The assumption $a_0 = 1$ can be always achieved by dividing all the coefficients by $a_0$ if $a_0 \neq 0$.
- The difference equation characterizes a recursive way of obtaining the output $y[n]$ from the input $x[n]$.
- When $a_k = 0$ for $k = 1 \ldots N$, the difference equation degenerates to a FIR (finite impulse response) system - the impulse response is of finite length.
  - The output consists of a linear combination of finite inputs.

$$y[n] = \sum_{m=0}^{M} b_m x[n - m]$$
Difference equation: IIR System

• When $b_m$ are not all zeros for $m = 1 \ldots M$, and $a_0 = 1$, the difference equation degenerates to

$$y[n] = \sum_{m=0}^{M} (b_m)x[n-m] + \sum_{k=1}^{N} (-a_k)y[n-k]$$

• This is an example of IIR (infinite impulse response) system
  – IIR system: systems with the impulse response being of infinite length.
Example

- Accumulator

\[ y[n] = \sum_{k=\infty}^{n} x[k] \]

\[ = x[n] + \sum_{k=\infty}^{n-1} x[k] = x[n] + y[n-1] \]
Moving average system when $M_1=0$:

- The impulse response is $h[n] = u[n] - u[n-M_2-1]$

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

- Also, note that

$$y[n] - y[n-1] = \frac{1}{M_2 + 1} (x[n] - x[n-M_2-1])$$

The term $y[n] - y[n-1]$ suggests the implementation can be cascaded with an accumulator.
Hence, there are at least two difference equation representations of the moving average system. First,

\[ x[n] \rightarrow b \rightarrow + \rightarrow y[n] \]

- where \( b = \frac{1}{(M+1)} \)
- and TD denotes one-sample delay

\[ x[n-1] \rightarrow b \rightarrow + \rightarrow y[n] \]

\[ x[n-2] \rightarrow b \rightarrow + \rightarrow y[n] \]

\[ x[n-M] \rightarrow b \rightarrow + \rightarrow y[n] \]
Moving Average System (continue)

- Second,

\[
\frac{1}{(M_2 + 1)}
\]

\[x[n]\]

\[x_1[n]\]

\[y[n]\]

- The first representation is FIR, and the second is IIR.
Solution of Difference Equations

- Just as differential equations for continuous-time systems, a linear constant-coefficient difference equation for discrete-time systems does not provide a unique solution if no additional constraints are provided.

- Solution: \( y[n] = y_p[n] + y_h[n] \)
  - \( y_h[n] \): homogeneous solution obtained by setting all the inputs as zeros.
    \[
    \sum_{k=1}^{N} a_k y[n-k] = 0
    \]
  - \( y_p[n] \): a particular solution satisfying the difference equation.
Solution of Difference Equations (continue)

- Additional constraints: consider the $N$ auxiliary conditions that $y[-1], y[-2], \ldots, y[-N]$ are given.
  - The other values of $y[n]$ ($n \geq 0$) can be generated by
    \[
y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m]
    \]
    when $x[n]$ is available, $y[1], y[2], \ldots y[n], \ldots$ can be computed recursively.
  - To generate values of $y[n]$ for $n < -N$ recursively,
    \[
y[n-N] = -\sum_{k=1}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{m=0}^{M} \frac{b_k}{a_N} x[n-m]
    \]
Consider the difference equation

\[ y[n] = a \cdot y[n-1] + x[n]. \]

- Assume the input is \( x[n] = K \cdot \delta[n] \), and the auxiliary condition is \( y[-1] = c \).
- Hence, \( y[0] = ac + K, y[1] = a \cdot y[0] + 0 = a^2c + aK, \ldots \)
- Recursively, we found that \( y[n] = a^{n+1}c + a^nK \), for \( n \geq 0 \).
- For \( n < -1 \), \( y[-2] = a^{-1}(y[-1] - x[-1]) = a^{-1}c \), \( y[-2] = a^{-1}y[-1] = a^{-2}c \), \ldots, and \( y[n] = a^{n+1}c \) for \( n < -1 \).
- Hence, the solution is

\[ y[n] = a^{n+1}c + Ka^n u[n], \]
Example of the Solutions (continue)

- The solution system is non-linear:
  - When $K=0$, i.e., the input is zero, the solution (system response) $y[n] = a^{n+1}c$.
  - Since a linear system requires that the output is zero for all time when the input is zero for all time.

- The solution system is not shift invariant:
  - when input were shifted by $n_0$ samples, $x_1[n] = K \cdot \delta[n - n_0]$, the output is $y_1[n] = a^{n+1}c + Ka^{n-n_0}u[n - n_0]$.

- The recursively-implemented system for finding the solution is non-causal.
LTI solution of difference equations

- Our principal interest in the text is in systems that are linear and time invariant.
- How to make the recursively-implemented solution system be LTI?
- Initial-rest condition:
  - If the input $x[n]$ is zero for $n$ less than some time $n_0$, the output $y[n]$ is also zero for $n$ less than $n_0$.
    - The previous example does not satisfy this condition since $x[n] = 0$ for $n<0$ but $y[-1] = c$.
- Property: If the initial-rest condition is satisfied, then the system will be LTI and causal.
Eigen function of a LTI system
- When applying an eigenfunction as input, the output is the same function multiplied by a constant.

$x[n] = e^{jwn}$ is the eigenfunction of all LTI systems.
- Let $h[n]$ be the impulse response of an LTI system, when $e^{jwn}$ is applied as the input,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{jw(n-k)} = e^{jwn} \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}$$
Eigenfunction of LTI

- Let $H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}$

  we have $y[n] = H(e^{jw}) e^{jwn}$

- Consequently, $e^{jwn}$ is the eigenfunction of the system, and the associated eigenvalue is $H(e^{jw})$.

- Remember that $H(e^{jw})$ is the DTFT of $h[n]$.

- We call $H(e^{jw})$ the LTI system’s frequency response
  - consisting of the real and imaginary parts, $H(e^{jw}) = H_R(e^{jw}) + jH_I(e^{jw})$, or in terms of magnitude and phase.
Example of Frequency Response

• Frequency response of the ideal delay system,
  \[ y[n] = x[n - n_d], \]

• If we consider \( x[n] = e^{jwn} \) as input, then
  \[ y[n] = e^{jw(n-n_d)} = e^{-jwn_d} e^{jwn} \]

  Hence, the frequency response is
  \[ H(e^{jw}) = e^{-jwn_d} \]

• The magnitude and phase are
  \[ |H(e^{jw})| = 1, \quad \angle H(e^{jw}) = -wn_d \]
• When a signal can be represented as a linear combination of complex exponentials (Fourier Series):

$$x[n] = \sum_{k} \alpha_k e^{j\omega_k n}$$

By the principle of superposition, the output is

$$y[n] = \sum_{k} \alpha_k H(e^{j\omega_k}) e^{j\omega_k n}$$

• Thus, we can find the output of linearly combined signals if we know the frequency response of the system.
Example of Linear Combination

- Sinusoidal responses of LTI systems:

\[ x[n] = A \cos(w_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{jw_0 n} + \frac{A}{2} e^{-j\phi} e^{-jw_0 n} \equiv x_1[n] + x_2[n] \]

- The response of \( x_1[n] \) and \( x_2[n] \) are

\[ y_1[n] = H(e^{jw_0}) \left( \frac{A}{2} \right) e^{j\phi} e^{jw_0 n} \]
\[ y_2[n] = H(e^{-jw_0}) \left( \frac{A}{2} \right) e^{-j\phi} e^{-jw_0 n} \]

- If \( h[n] \) is real, by the DTFT property that \( H(e^{-jw_0}) = H^*(e^{jw_0}) \), the total response \( y[n] = y_1[n] + y_2[n] \) is

\[ y[n] = A \left| H(e^{jw_0}) \right| \cos(w_0 n + \phi + \theta), \text{ where } \theta = \angle H(e^{jw_0}) \]
Difference to Continuous-time System Response

- For a continuous-time system, the frequency response applied is the continuous Fourier transform, which is not necessarily to be periodic.

- However, for a discrete-time system, the frequency response is always periodic with period $2\pi$, since

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk} = \sum_{k=-\infty}^{\infty} h[k] e^{-j(w+2\pi)k} = H(e^{j(w+2\pi)})$$

- Because $H(e^{jw})$ is periodic with period $2\pi$, we need only specify $H(e^{jw})$ over an interval of length $2\pi$, eg., $[0, 2\pi]$ or $[-\pi, \pi]$. For consistency, we choose the interval $[-\pi, \pi]$.

- The inherent periodicity defines the frequency response everywhere outside the chosen interval.
Convolution vs. Multiplication

• For DTFT, when performing convolution in time domain, it is equivalent to perform multiplication in the frequency domain.

• Hence, for an LTI system with the impulse response being $h[n]$, when the input is $x[n]$
  – We know that $y[n] = h[n]*x[n]$.
  – The spectrum of $y[n]$ shall be $Y(e^{jw}) = H(e^{jw})X(e^{jw})$.
  – i.e., the spectrum of $y[n]$ can be obtained by multiplying the spectrum of $x[n]$ with the frequency response.
The “low frequencies” are frequencies close to zero, while the “high frequencies” are those close to $\pm \pi$. Since that the frequencies differing by an integer multiple of $2\pi$ are indistinguishable, the “low frequency” are those that are close to an even multiple of $\pi$, while the “high frequencies” are those close to an odd multiple of $\pi$.

Ideal frequency-selective filters:

- An important class of linear-invariant systems includes those systems for which the frequency response is unity over a certain range of frequencies and is zero at the remaining frequencies.
Frequency Response of **Ideal Low-pass Filter**
Frequency Response of Ideal High-pass Filter

\[ H_{hp}(e^{j\omega}) \]

- \( \pi \)
- \( -\omega_c \)
- \( 0 \)
- \( \omega_c \)
- \( \pi \)
- \( \omega \)
Frequency Response of Ideal Band-stop Filter
Frequency Response of Ideal Band-pass Filter
Frequency Response of the Moving-average System

- The impulse response of the moving-average system is

$$h[n] = \begin{cases} 
\frac{1}{M_1 + M_2 + 1} & -M_1 \leq n \leq M_2 \\
0 & \text{otherwise}
\end{cases}$$

- Therefore, the frequency response is

$$H \left[ e^{jw} \right] = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-jwn}$$

- By noting that the following formula holds:

$$\sum_{k=n}^{m} \alpha^k = \frac{\alpha^n - \alpha^{m+1}}{1-\alpha}, \quad m > n$$
Frequency Response of the Moving-average System (continue)

\[
H[e^{jw}] = \frac{1}{M_1 + M_2 + 1} \frac{e^{jwM_1} - e^{-jw(M_2+1)}}{1 - e^{-jw}}
\]

\[
= \frac{1}{M_1 + M_2 + 1} \frac{e^{jw(M_1+M_2+1)/2} - e^{-jw(M_1+M_2+1)/2}}{1 - e^{-jw}} e^{-jw(M_2-M_1+1)/2}
\]

\[
= \frac{1}{M_1 + M_2 + 1} \frac{e^{jw(M_1+M_2+1)/2} - e^{-jw(M_1+M_2+1)/2}}{e^{jw/2} - e^{-jw/2}} e^{-jw(M_2-M_1)/2}
\]

\[
= \frac{1}{M_1 + M_2 + 1} \frac{\sin\left[jw(M_1 + M_2 + 1)/2\right]}{\sin(w/2)} e^{-jw(M_2-M_1)/2}
\]

\[
= |H(e^{jw})| \exp^{-j\angle H(e^{jw})} \quad \text{(magnitude and phase)}
\]
Frequency Response of the Moving-average System (continue)

\[ M_1 = 0 \text{ and } M_2 = 4 \]

Amplitude response

Phase response
Example

- Determining the impulse response for a difference equation

\[ y[n] - (1/2) y[n-1] = x[n] - (1/4)x[n-1] \]

To find the impulse response, we set \( x[n] = \delta[n] \). Then the above equation becomes

\[ h[n] - (1/2) h[n-1] = \delta[n] - (1/4)\delta[n-1] \]

Applying the Fourier transform, we obtain

\[ H(e^{jw}) - (1/2)e^{jw}H(e^{jw}) = 1 - (1/4) e^{-jw} \]

So \( H(e^{jw}) = (1 - (1/4) e^{-jw}) / (1 - (1/2) e^{-jw}) \)
Example (continue)

- To obtain the impulse response $h[n]$
- From the DTFT pair-wise table, we know that

$$a^n u[n] \ (|a| < 1) \leftrightarrow \frac{1}{1 - ae^{-jw}}$$

thus, $(1/2)^n u[n] \leftrightarrow 1 / (1 - (1/2) e^{-jw})$

By the shifting property,

$$-(1/4)(1/2)^{n-1} u[n-1] \leftrightarrow -(1/4) e^{-jw} / (1 - (1/2) e^{-jw})$$

Thus,

$$h[n] = (1/2)^n u[n] - (1/4)(1/2)^{n-1} u[n-1]$$
Suddenly Applied Complex Exponential Inputs

• In practice, we may not apply the complex exponential inputs $e^{jwn}$ to a system, but the more practical-appearing inputs of the form

$$x[n] = e^{jwn} \cdot u[n]$$

  - i.e., complex exponentials that are suddenly applied at an arbitrary time, which for convenience we choose $n=0$.
  - Consider its output to a causal LTI system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \begin{cases} 
0 & \text{if } n < 0 \\
\left( \sum_{k=0}^{n} h[k] e^{-jwk} \right) e^{jwn} & \text{if } n \geq 0
\end{cases}$$
Suddenly Applied Complex Exponential Inputs (continue)

• We consider the output for $n \geq 0$.

$$y[n] = \left( \sum_{k=0}^{\infty} h[k] e^{-jwk} \right) e^{jwn} - \left( \sum_{k=n+1}^{\infty} h[k] e^{-jwk} \right) e^{jwn}$$

$$= H(e^{jw}) e^{jwn} - \left( \sum_{k=n+1}^{\infty} h[k] e^{-jwk} \right) e^{jwn}$$

- Hence, the output can be written as $y[n] = y_{ss}[n] + y_t[n]$, where

$$y_{ss}[n] = H(e^{jw}) e^{jwn} \quad \text{Steady-state response}$$

$$y_t[n] = - \sum_{k=n+1}^{\infty} h[k] e^{-jwk} e^{jwn} \quad \text{Transient response}$$
Suddenly Applied Complex Exponential Inputs (continue)

- If \( h[n] = 0 \) except for \( 0 \leq n \leq M \) (i.e., a FIR system), then the transient response \( y_t[n] = 0 \) for \( n+1 > M \). That is, the transient response becomes zero since the time \( n = M \). For \( n \geq M \), only the steady-state response exists.

- For infinite-duration impulse response (i.e., IIR)

\[
|y_t[n]| = \left| \sum_{k=n+1}^{\infty} h[k] e^{-jwk} e^{jwn} \right| \leq \sum_{k=n+1}^{\infty} |h[k]| \equiv Q_n
\]

- For stable system, \( Q_n \) must become increasingly smaller as \( n \to \infty \), and so is the transient response.
Suddenly Applied Complex Exponential Inputs (continue)

Illustration for the FIR case by convolution
Suddenly Applied Complex Exponential Inputs (continue)

Illustration for the IIR case by convolution