Sampling and Reconstruction

The impulse response of a continuous-time ideal low pass filter is the inverse continuous Fourier transform of its frequency response.

Let $H_{lp}(j\omega)$ be the frequency response of the ideal low-pass filter with the cut-off frequency being $\omega_s/2$.

\[
h_{lp}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{lp}(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} T e^{j\omega t} d\omega = \sin\left(\frac{\omega_s t}{2}\right) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\left(\frac{\omega_s t}{2}\right)}
\]
Remember that when we sample a continuous band-limited signal satisfying the sampling theorem, then the signal can be reconstructed by ideal low-pass filtering.

\[ s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \]

\[ x_c(t) \times H_r(j\Omega) \rightarrow x_r(t) \]

\[ X_c(j\Omega) \]

\[ X_s(j\Omega) \]

\[ X_r(j\Omega) \]

\[ H_r(j\Omega) \]

\( \Omega_N < \Omega_c < (\Omega_s - \Omega_N) \)

\( \Omega_s > 2\Omega_N \)
The sampled continuous-time signal can be represented by an impulse train:

\[ x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) \]

Note that when we input the impulse function \( \delta(t) \) into the ideal low-pass filter, its output is its impulse response

\[ h_r(t) \equiv h_{lp}(t) = \sin(\pi t / T) / (\pi t / T) \]

When we input \( x_s(t) \) to an ideal low-pass filter, the output shall be

\[ x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T} \]

- It shows that how the continuous-time signal can be reconstructed by interpolating the discrete-time signal \( x[n] \).
Properties:
\[ h_r(0) = 1; \quad h_r(nT) = 0 \text{ for } n=\pm1, \pm2, \ldots; \]
It follows that \( x_r(mT) = x_c(mT) \), for all integer \( m \).

The form \( \sin(t)/t \) is referred to as a sinc function. So, the interpolant of ideal low-pass function is a sinc function.
Illustration of reconstruction

\[ x_c(t) \]

\[ x_r(t) \]

\[ T \]
Ideal discrete-to-continuous (D/C) converter

- It defines an ideal system for reconstructing a bandlimited signal from a sequence of samples.
Ideal low pass filter is non-causal

- Since its impulse response is not zero when n<0.

Ideal low pass filter can not be realized

- It can not be implemented by using any difference equations.

Hence, in practice, we need to design filters that can be implemented by difference equations, to approximate ideal filters.
Z-Transform

- Discrete-time Fourier Transform

\[ X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \]

- z-transform: polynomial representation of a sequence

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]
Z-Transform (continue)

- Z-transform operator:  \[ Z\{ \cdot \} \]

\[
Z\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]z^{-k} = X(z)
\]

- The z-transform operator is seen to transform the sequence \(x[n]\) into the function \(X\{z\}\), where \(z\) is a continuous complex variable.
  - From time domain (or space domain, n-domain) to the z-domain

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \leftrightarrow \quad Z\{x[n]\} = X(Z)
\]
Bilateral vs. Unilateral

• Two sided or bilateral z-transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

• Unilateral z-transform

\[ X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \]
Example of z-transform

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n \leq -1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$N &gt; 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$
Relationship to the Fourier Transform

• If we replace the complex variable \( z \) in the \( z \)-transform by \( e^{jw} \), then the \( z \)-transform reduces to the Fourier transform.

• The Fourier transform is simply the \( z \)-transform when evaluating \( X(z) \) only in a unit circle in the \( z \)-plane.

• From another point of view, we can express the complex variable \( z \) in the polar form as \( z = re^{jw} \). With \( z \) expressed in this form,

\[
X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n](re^{jw})^{-n} = \sum_{n=-\infty}^{\infty} \left( x[n]r^{-n} \right)e^{-jwn}
\]
In this sense, the z-transform can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence $r^{-n}$.

- For $r=1$, the z-transform reduces to the Fourier transform.

The unit circle in the complex $z$ plane

The unit circle in the complex $z$ plane
Relationship to the Fourier Transform (continue)

- Beginning at \( z = 1 \) (i.e., \( w = 0 \)) through \( z = j \) (i.e., \( w = \pi/2 \)) to \( z = -1 \) (i.e., \( w = \pi \)), we obtain the Fourier transform from \( 0 \leq w \leq \pi \).
- Continuing around the unit circle in the \( z \)-plane corresponds to examining the Fourier transform from \( w = \pi \) to \( w = 2\pi \).
- Fourier transform is usually displayed on a linear frequency axis. Interpreting the Fourier transform as the \( z \)-transform on the unit circle in the \( z \)-plane corresponds conceptually to wrapping the linear frequency axis around the unit circle.
Convergence Region of Z-transform

- The sum of the series may not be converge for all $z$.

- **Region of convergence (ROC)**
  - Since the z-transform can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence $r^{-n}$, it is possible for the z-transform to converge even if the Fourier transform does not.

  \[
  X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \leq \sum_{n=-\infty}^{\infty} |x[n]|z^{-n}
  \]

  - Eg., $x[n] = u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with ROC $|z| > 1$. 

ROC of Z-transform

• In fact, convergence of the power series $X(z)$ depends only on $|z|$.  
$$\sum_{n=-\infty}^{\infty} |x[n]| |z^{-n}| < \infty$$

• If some value of $z$, say $z = z_1$, is in the ROC, then all values of $z$ on the circle defined by $|z| = |z_1|$ will also be in the ROC.

• Thus the ROC will consist of a ring in the z-plane.
ROC of Z-transform – Ring Shape
Analytic Function and ROC

• The z-transform is a Laurent series of $z$.
  - A number of theorems from the complex-variable theory can be employed to study the z-transform.
  - A Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence.
  - Hence, the z-transform and all its derivatives exist and must be continuous functions of $z$ with the ROC.
  - This implies that if the ROC includes the unit circle, the Fourier transform and all its derivatives with respect to $w$ must be continuous function of $w$. 
Z-transform and Linear Systems

- First, we analyze the Z-transform of a causal FIR system

\[ y[n] = \sum_{m=0}^{M} b_m x[n-m] \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n&lt;0 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>( M )</th>
<th>( N&gt; M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h[n] )</td>
<td>0</td>
<td>( b_0 )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>...</td>
<td>( b_M )</td>
<td>0</td>
</tr>
</tbody>
</table>

- The impulse response is

\[ h[n] = \sum_{m=0}^{M} b_m \delta[n-m] \]

- Take the z-transform on both sides
Z-transform of Causal FIR System (continue)

\[ Y(z) = Z\{y[n]\} = Z\left\{ \sum_{m=0}^{M} b_m x[n-m] \right\} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M} b_m x[n-m]z^{-n} \]

\[ = \sum_{m=0}^{M} b_m \sum_{n=-\infty}^{\infty} x[n-m]z^{-n} = \sum_{m=0}^{M} b_m z^{-m} \left( \sum_{n=-\infty}^{\infty} x[n-m]z^{-(n-m)} \right) \]

\[ = \sum_{m=0}^{M} b_m z^{-m} Z\{x[n]\} = Z\{h[n]\}X[z] \equiv H(z)X(z) \]

• Thus, the z-transform of the output of a FIR system is the product of the z-transform of the input signal and the z-transform of the impulse response.
Z-transform of Causal FIR System (continue)

\[ H(z) = \sum_{m=0}^{M} b_m z^{-m} \]

- \( H(z) \) is called the **system function** (or **transfer function**) of a (FIR) LTI system.
Multiplication Rule of Cascading System

\[ X(z) \xrightarrow{H_1(z)} Y(z) \xrightarrow{Y(z)} H_2(z) \xrightarrow{V(z)} \]

\[ \equiv \]

\[ X(z) \xrightarrow{H_1(z)} Y(z) \xrightarrow{H_2(z)} V(z) \]

\[ \equiv \]

\[ X(z) \xrightarrow{H_1(z)} H_1(z)H_2(z) \xrightarrow{Y(z)} \]
Example

• Consider the FIR system $y[n] = 6x[n] - 5x[n-1] + x[n-2]$

• The z-transform system function is

$$H(z) = 6 - 5z^{-1} + z^{-2}$$

$$= (3 - z^{-1})(2 - z^{-1}) = 6\frac{(z - \frac{1}{3})(z - \frac{1}{2})}{z^2}$$
Delay of one Sample

• Consider the FIR system $y[n] = x[n-1]$, i.e., the one-sample-delay system.

• The z-transform system function is

\[ H(z) = z^{-1} \]
Delay of k Samples

• Similarly, the FIR system $y[n] = x[n-k]$, i.e., the k-sample-delay system, is the z-transform of the impulse response $\delta[n - k]$.

\[ H(z) = z^{-k} \]
System Diagram of A Causal FIR System

- The signal-flow graph of a causal FIR system can be re-represented by z-transforms.
Z-transform of General Difference Equation (IIR system)

- Remember that the general form of a linear constant-coefficient difference equation is

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \quad \text{for all } n \]

- When \( a_0 \) is normalized to \( a_0 = 1 \), the system diagram can be shown as below
Review of Linear Constant-coefficient Difference Equation

\[ x[n] \rightarrow b_0 \rightarrow + \rightarrow + \rightarrow \ldots \rightarrow -a_1 \rightarrow -a_2 \rightarrow \ldots \rightarrow -a_N \rightarrow y[n] \]

\[ x[n-1] \rightarrow b_1 \rightarrow + \rightarrow + \rightarrow \ldots \rightarrow y[n-1] \]

\[ x[n-2] \rightarrow b_2 \rightarrow + \rightarrow + \rightarrow \ldots \rightarrow y[n-2] \]

\[ x[n-M] \rightarrow b_M \rightarrow + \rightarrow + \rightarrow \ldots \rightarrow y[n-N] \]
Z-transform of Linear Constant-coefficient Difference Equation

• The signal-flow graph of difference equations represented by z-transforms.

\[ X(z) \rightarrow b_0 \rightarrow + \rightarrow + \rightarrow + \rightarrow + \rightarrow Y(z) \]

\[ z^{-1} \]

\[ b_1 \]

\[ z^{-1} \]

\[ b_2 \]

\[ \ldots \]

\[ z^{-1} \]

\[ b_M \]

\[ -a_1 \]

\[ z^{-1} \]

\[ -a_2 \]

\[ \ldots \]

\[ z^{-1} \]

\[ -a_N \]
Z-transform of Difference Equation (continue)

- From the signal-flow graph,

\[ Y(z) = \sum_{m=0}^{M} b_m X(z)z^{-m} - \sum_{k=1}^{N} a_k Y(z)z^{-k} \]

- Thus,

\[ \sum_{k=0}^{N} a_k Y(z)z^{-k} = \sum_{m=0}^{M} b_m X(z)z^{-m} \]

- We have

\[ \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}} \]
Let \( H(z) \) is called the system function of the LTI system defined by the linear constant-coefficient difference equation.

- The multiplication rule still holds: \( Y(z) = H(z)X(z) \), i.e.,
  \[
  Z\{y[n]\} = H(z)Z\{x[n]\}.
  \]

- The system function of a difference equation (or a generally IIR system) is a rational form \( X(z) = P(z)/Q(z) \).

- Since LTI systems are often realized by difference equations, the rational form is the most common and useful for z-transforms.
Z-transform of Difference Equation (continue)

- When $a_k = 0$ for $k = 1 \ldots N$, the difference equation degenerates to a FIR system we have investigated before.

$$H(z) = \sum_{m=0}^{M} b_m z^{-m}$$

- It can still be represented by a rational form of the variable $z$ as

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{M-m}}{z^M}$$
System Function and Impulse Response

- When the input $x[n] = \delta[n]$, the z-transform of the impulse response satisfies the following equation:
  \[
  Z\{h[n]\} = H(z)Z\{\delta[n]\}.
  \]

- Since the z-transform of the unit impulse $\delta[n]$ is equal to one, we have
  \[
  Z\{h[n]\} = H(z)
  \]

- That is, the system function $H(z)$ is the z-transform of the impulse response $h[n]$. 
Z-transform vs. Convolution

- Convolution

\[ x(n) \ast h(n) = h(n) \ast x(n) \]

\[ y(n) = \sum_{k=0}^{\infty} x(k)h(n - k). \]

- Take the z-transform on both sides:

\[ Y(z) = \sum_{n=0}^{\infty} y(n)z^{-n} = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{\infty} x(k)h(n - k) \right] z^{-n} \]

Interchanging the order of summation, we obtain

\[ Y(z) = \sum_{k=0}^{\infty} x(k) \sum_{n=0}^{\infty} h(n - k)z^{-n} \]
Let us make a substitution \( m = n - k \), and now we have

\[
Y(z) = \sum_{k=0}^{\infty} x(k) \sum_{m=-k}^{\infty} h(m)z^{-(m+k)}
\]

\[
= \sum_{k=0}^{\infty} x(k)z^{-k} \sum_{m=-k}^{\infty} h(m)z^{-m}
\]

But \( h(m) = 0 \) for \(-k \leq m \leq -1\), so that

\[
Y(z) = \sum_{k=0}^{\infty} x(k)z^{-k} \sum_{m=0}^{\infty} h(m)z^{-m}
\]

\[
= X(z)H(z) = H(z)X(z)
\]

Time domain convolution implies Z-domain multiplication
Generally, for a linear system,
\[ y[n] = T\{x[n]\} \]
- it can be shown that
\[ Y\{z\} = H(z)X(z). \]
where \(H(z)\), the system function, is the z-transform of the impulse response of this system \(T\{\cdot\}\).
- Also, cascading of systems becomes multiplication of system function under z-transforms.

\[ \begin{align*}
X(z) &\quad \xrightarrow{H(z)/H(e^{jw})} \quad Y(z) \quad (= H(z)X(z)) \\
X(e^{jw}) &\quad \xrightarrow{H(e^{jw})} \quad Y(e^{jw}) \quad (= H(e^{jw})X(e^{jw}))
\end{align*} \]

Z-transform
Fourier transform
Poles and Zeros

- **Pole:**
  - The *pole* of a z-transform $X(z)$ are the values of $z$ for which $X(z) = \infty$.

- **Zero:**
  - The *zero* of a z-transform $X(z)$ are the values of $z$ for which $X(z) = 0$.

- When $X(z) = P(z)/Q(z)$ is a rational form, and both $P(z)$ and $Q(z)$ are polynomials of $z$, the poles of $X(z)$ are the roots of $Q(z)$, and the zeros are the roots of $P(z)$, respectively.
Examples

• Zeros of a system function
  - The system function of the FIR system $y[n] = 6x[n] - 5x[n-1] + x[n-2]$ has been shown as
    
    $$H(z) = 6 \frac{(z - \frac{1}{3})(z - \frac{1}{2})}{z^2} = \frac{P(z)}{Q(z)}$$

• The zeros of this system are 1/3 and 1/2, and the pole is 0.
• Since 0 and 0 are double roots of $Q(z)$, the pole is a second-order pole.
Example: Finite-length Sequence (FIR System)

Given \( x(n) = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \)

Then
\[
X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}}
\]

\[
= \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}
\]

There are the \( N \) roots of \( z^N = a^N \), \( z_k = ae^{j(2\pi k/N)} \). The root of \( k = 0 \) cancels the pole at \( z = a \). Thus there are \( N-1 \) zeros, \( z_k = ae^{j(2\pi k/N)} \), \( k = 1 \ldots N \), and a \( (N-1) \)th order pole at zero.
Pole-zero Plot
Example: Right-sided Exponential Sequence

- Right-sided sequence:
  - A discrete-time signal is right-sided if it is nonzero only for \( n \geq 0 \).

- Consider the signal \( x[n] = a^n u[n] \).

\[
X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n
\]

- For convergent \( X(z) \), we need \( \sum_{n=0}^{\infty} (az^{-1})^n < \infty \)

  - Thus, the ROC is the range of values of \( z \) for which \( |az^{-1}| < 1 \) or, equivalently, \( |z| > a \).
Example: Right-sided Exponential Sequence (continue)

- By sum of power series,
  \[ X(z) = \sum_{n=0}^{\infty} \left( a z^{-1} \right)^n = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}, \quad |z| > |a| \]

- There is one zero, at \( z=0 \), and one pole, at \( z=a \).
Example: Left-sided Exponential Sequence

• Left-sided sequence:
  - A discrete-time signal is left-sided if it is nonzero only for $n \leq -1$.

• Consider the signal $x[n] = -a^n u[-n-1]$.

$$\begin{align*}
X(z) &= - \sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = - \sum_{n=-\infty}^{-1} a^n z^{-n} \\
&= \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n
\end{align*}$$

  - If $|az^{-1}| < 1$ or, equivalently, $|z| < a$, the sum converges.
Example: Left-sided Exponential Sequence (continue)

- By sum of power series,
  \[ X(z) = 1 - \frac{1}{1 - a^{-1}z} = -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{z}{z-a}, \quad |z| < |a| \]

- There is one zero, at \( z=0 \), and one pole, at \( z=a \).

The pole-zero plot and the algebraic expression of the system function are the same as those in the previous example, but the ROC is different.
Hence, to uniquely identify a sequence from its z-transform, we have to specify additionally the ROC of the z-transform.

Another example: given \( x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n) \)

Then \( X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u(n)z^{-n} \)

\[ = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n} \]

\[ = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2z\left(z - \frac{1}{12}\right)}{(z - \frac{1}{2})(z + \frac{1}{3})} \]
\[
\left(\frac{1}{2}\right)^n u(n) \quad \leftrightarrow \quad \frac{z}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}
\]

\[
\left(-\frac{1}{3}\right)^n u(n) \quad \leftrightarrow \quad \frac{z}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}
\]

Thus

\[
\left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n) \quad \leftrightarrow \quad \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{2}
\]
The images display three different complex plane diagrams labeled as $\mathcal{I}m$ z-plane. Each diagram contains circles and marked points:

- The first diagram has a circle with a mark at $\frac{1}{2}$.
- The second diagram has a circle with marks at $\frac{1}{3}$ and $\frac{1}{2}$.
- The third diagram has a circle with marks at $\frac{1}{3}$, $\frac{1}{12}$, and $\frac{1}{2}$.
Consider another two-sided exponential sequence

Given \[ x(n) = \left( -\frac{1}{3} \right)^n u(n) - \left( \frac{1}{2} \right)^n u(-n-1) \]

Since \[ \left( -\frac{1}{3} \right)^n u(n) \leftrightarrow \frac{1}{1 + \frac{1}{3} z^{-1}}, \quad |z| > \frac{1}{3} \]

and by the left-sided sequence example

\[ -\left( \frac{1}{2} \right)^n u(-n-1) \leftrightarrow \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| < \frac{1}{2} \]
\[ X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{2z\left(z - \frac{1}{12}\right)}{(z + \frac{1}{3})(z - \frac{1}{2})} \]

Again, the poles and zeros are the same as the previous example, but the ROC is not.
Some Common Z-transform Pairs

\[ \delta[n] \leftrightarrow 1 \quad \text{ROC: all } z. \]

\[ u[n] \leftrightarrow \frac{1}{1-z^{-1}} \quad \text{ROC: } |z| > 1. \]

\[ -u[-n-1] \leftrightarrow \frac{1}{1-z^{-1}} \quad \text{ROC: } |z| < 1. \]

\[ \delta[n-m] \leftrightarrow z^{-m} \quad \text{ROC: all } z \text{ except } 0 \text{ (if } m > 0 \text{) or } \infty \text{ (if } m < 0 \text{).} \]

\[ a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}} \quad \text{ROC: } |z| > |a|. \]

\[ -a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}} \quad \text{ROC: } |z| < |a|. \]

\[ na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \quad \text{ROC: } |z| > |a|. \]
Some Common Z-transform Pairs (continue)

\[-na^n u[-n-1] \Leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}\]  
\[\text{ROC : } |z| < |a|.\]

\[[\cos w_0 n] u[n] \Leftrightarrow \frac{1-\cos w_0 z^{-1}}{1-2\cos w_0 z^{-1} + z^{-2}}\]  
\[\text{ROC : } |z| > 1.\]

\[[\sin w_0 n] u[n] \Leftrightarrow \frac{\sin w_0 z^{-1}}{1-2\cos w_0 z^{-1} + z^{-2}}\]  
\[\text{ROC : } |z| > 1.\]

\[[r^n \cos w_0 n] u[n] \Leftrightarrow \frac{1-\cos w_0 z^{-1}}{1-2r \cos w_0 z^{-1} + r^2 z^{-2}}\]  
\[\text{ROC : } |z| > r.\]

\[[r^n \sin w_0 n] u[n] \Leftrightarrow \frac{r \sin w_0 z^{-1}}{1-2r \cos w_0 z^{-1} + r^2 z^{-2}}\]  
\[\text{ROC : } |z| > r.\]

\[\begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}}\]  
\[\text{ROC : } |z| > 0.\]
Properties of the ROC

• The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \leq r_R < \vert z \vert \leq r_L \leq \infty$.

• The Fourier transform of $x[n]$ converges absolutely iff the ROC includes the unit circle.

• The ROC cannot contain any poles

• If $x[n]$ is a finite-length (finite-duration) sequence, then the ROC is the entire z-plane except possible $z = 0$ or $z = \infty$.

• If $x[n]$ is a right-sided sequence, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly include) $z = \infty$. 
Properties of the ROC (continue)

- If $x[n]$ is a left-sided sequence, the ROC extends inward from the innermost (i.e., smallest magnitude) nonzero pole in $X(z)$ to (and possibly include) $z = 0$.
- A two-sided sequence $x[n]$ is an infinite-duration sequence that is neither right nor left sided. The ROC will consist of a ring in the $z$-plane, bounded on the interior and exterior by a pole, but not containing any poles.
- The ROC must be a connected region.
Example

A system with three poles
Different possibilities of the ROC. (b) ROC to a right-sided sequence. (c) ROC to a left-handed sequence.
Different possibilities of the ROC. (b) ROC to a two-sided sequence. (c) ROC to another two-sided sequence.
ROC vs. Linear System

- Consider the system function $H(z)$ of a linear system:
  - If the system is stable, the impulse response $h(n)$ is absolutely summable and therefore has a Fourier transform, then the ROC must include the unit circle.
  - If the system is causal, then the impulse response $h(n)$ is right-sided, and thus the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $H(z)$ to (and possibly include) $z = \infty$. 
Inverse Z-transform

- Given $X(z)$, find the sequence $x[n]$ that has $X(z)$ as its z-transform.
- We need to specify both algebraic expression and ROC to make the inverse Z-transform unique.
- Techniques for finding the inverse z-transform:
  - Investigation method:
    - By inspect certain transform pairs.
    - Eg. If we need to find the inverse z-transform of
      $$X(z) = \frac{1}{1 - 0.5z^{-1}}$$
      From the transform pair we see that $x[n] = 0.5^n u[n]$. 

Inverse Z-transform by Partial Fraction Expansion

• If $X(z)$ is the rational form with

$$X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

• An equivalent expression is

$$X(z) = \frac{z^{-M} \sum_{m=0}^{M} b_m z^{M-m}}{z^{-N} \sum_{k=0}^{N} a_k z^{N-k}} = \frac{z^{N} \sum_{m=0}^{M} b_m z^{M-m}}{z^{M} \sum_{k=0}^{N} a_k z^{N-k}}$$
Inverse Z-transform by Partial Fraction Expansion (continue)

- There will be $M$ zeros and $N$ poles at nonzero locations in the z-plane.
- Note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \prod_{m=1}^{M} \left(1 - c_m z^{-1}\right) \prod_{m=1}^{N} \left(1 - d_k z^{-1}\right)$$

where $c_k$'s and $d_k$'s are the nonzero zeros and poles, respectively.
Inverse Z-transform by Partial Fraction Expansion (continue)

- Then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Obviously, the common denominators of the fractions in the above two equations are the same. Multiplying both sides of the above equation by $1 - d_k z^{-1}$ and evaluating for $z = d_k$ shows that

$$A_k = \left(1 - d_k z^{-1}\right)X(z) \bigg|_{z=d_k}$$
Example

• Find the inverse z-transform of

\[ X(z) = \frac{1}{(1-(1/4)z^{-1})(1-(1/2)z^{-1})} \quad |z| > \frac{1}{2} \]

\( X(z) \) can be decomposed as

\[ X(z) = \frac{A_1}{1-(1/4)z^{-1}} + \frac{A_2}{1-(1/2)z^{-1}} \]

Then

\[ A_1 = \left(1-(1/4)z^{-1}\right)X(z)\bigg|_{z=1/4} = -1 \]
\[ A_2 = \left(1-(1/2)z^{-1}\right)X(z)\bigg|_{z=1/2} = 2 \]
Example (continue)

• Thus

\[ X(z) = \frac{-1}{1 - \left(\frac{1}{4}\right)z^{-1}} + \frac{2}{1 - \left(\frac{1}{2}\right)z^{-1}} \]

From the ROC, we have a right-hand sequence. So

\[ x[n] = 2 \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n] \]
Another Example

• Find the inverse z-transform of

\[ X(z) = \frac{(1+z^{-1})^2}{(1-(1/2)z^{-1})(1-z^{-1})} \quad |z| > 1 \]

Since both the numerator and denominator are of degree 2, a constant term exists.

\[ X(z) = B_0 + \frac{A_1}{1-(1/2)z^{-1}} + \frac{A_2}{1-z^{-1}} \]

\( B_0 \) can be determined by the fraction of the coefficients of \( z^{-2} \), \( B_0 = 1/(1/2) = 2 \).
Another Example (continue)

\[ X(z) = 2 + \frac{A_1}{1-(1/2)z^{-1}} + \frac{A_2}{1-z^{-1}} \]

\[ A_1 = 2 + \frac{-1+5z^{-1}}{1-(1/2)z^{-1}(1-z^{-1})} \left(1-(1/2)z^{-1}\right)_{z=1/2} = 9 \]

\[ A_2 = 2 + \frac{-1+5z^{-1}}{1-(1/2)z^{-1}(1-z^{-1})} \left(1-z^{-1}\right)_{z=1} = 8 \]

From the ROC, the solution is right-handed. So

\[ X(z) = 2 - \frac{9}{1-(1/2)z^{-1}} + \frac{8}{1-z^{-1}} \]

\[ x[n] = 2\delta[n] - 9(1/2)^n u[n] + 8u[n] \]
**Power Series Expansion**

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

\[ = ... + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + ... \]

- We can determine any particular value of the sequence by finding the coefficient of the appropriate power of \( z^{-1} \).
Example: Finite-length Sequence

- Find the inverse z-transform of

\[ X(z) = z^2 \left(1 - 0.5z^{-1}\right) \left(1 + z^{-1}\right) \left(1 - z^{-1}\right) \]

By directly expand \(X(z)\), we have

\[ X(z) = z^2 - 0.5z - 1 + 0.5z^{-1} \]

Thus,

\[ x[n] = \delta[n + 2] - 0.5\delta[n + 1] - \delta[n] + 0.5\delta[n - 1] \]
Example

- Find the inverse z-transform of

\[ X(z) = \log(1 + az^{-1}) \quad |z| > |a| \]

Using the power series expansion for \( \log(1+x) \) with \( |x| < 1 \), we obtain

\[ X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \]

Thus

\[ x[n] = \begin{cases} 
(-1)^{n+1} a^n / n & n \geq 1 \\
0 & n \leq 0
\end{cases} \]
Z-transform Properties

• Suppose

\[ x[n] \overset{z}{\leftrightarrow} X(z) \quad \text{ROC} = R_x \]
\[ x_1[n] \overset{z}{\leftrightarrow} X_1(z) \quad \text{ROC} = R_{x_1} \]
\[ x_2[n] \overset{z}{\leftrightarrow} X_2(z) \quad \text{ROC} = R_{x_2} \]

• Linearity

\[ ax_1[n] + b x_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z) \quad \text{ROC} = R_{x_1} \cap R_{x_2} \]
Z-transform Properties  
(continue)

- Time shifting

\[ x[n - n_0] \overset{z}{\leftrightarrow} z^{-n_0} X(z) \quad \text{ROC} = R_x \]  
(except for the possible addition or deletion of \( z=0 \) or \( z=\infty \).)

- Multiplication by an exponential sequence

\[ z^n_0 x[n] \overset{z}{\leftrightarrow} X(z / z_0) \quad \text{ROC} = |z_0|R_x \]
Z-transform Properties (continue)

- Differentiation of $X(z)$

$$nx[n] \overset{z}{\leftrightarrow} -z \frac{dX(z)}{dz} \quad \text{ROC} = R_x$$

- Conjugation of a complex sequence

$$x^*[n] \overset{z}{\leftrightarrow} X^*(z^*) \quad \text{ROC} = R_x$$
Z-transform Properties (continue)

• Time reversal

\[ x^*[−n] \leftrightarrow X^*(1/z^*) \quad \text{ROC} = \frac{1}{R_x} \]

If the sequence is real, the result becomes

\[ x[−n] \leftrightarrow X(1/z) \quad \text{ROC} = \frac{1}{R_x} \]

• Convolution

\[ x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \quad \text{ROC contains } R_{x_1} \cap R_{x_2} \]
Z-transform Properties (continue)

- Initial-value theorem: If \( x[n] \) is zero for \( n<0 \) (i.e., if \( x[n] \) is causal), then

\[
x[0] = \lim_{z \to \infty} X(z)
\]

- Inverse z-transform formula:

\[
Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}
\]

\[
Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint X(z)z^{n-1} dz
\]

Find the inverse z-transform by Integration in the complex domain.
1. A finite-length sequence is non-zero only at a finite number of positions. If $m$ and $n$ are the first and last non-zero positions, respectively, then we call $n-m+1$ the length of that sequence. What maximum length can the result of the convolution of two sequences of length $k$ and $l$ have?

2. An LTI system is described by the following difference equation: $y[n] = 0.3y[n-1] + y[n-2] - 0.2y[n-3] + x[n]$. Find the system function and frequency response of this system.

3. Find the inverse z-transform of the following system:

$$Y(z) = \frac{z(z-1)}{(z+1)(z + \frac{1}{3})} \quad \text{for } |z| < \frac{1}{3}$$