Geometric Modeling

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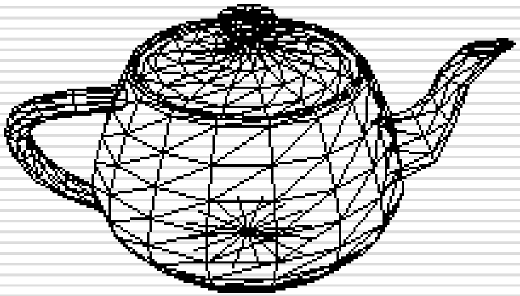
Parametric Curves and Surfaces

- Mathematical Curve Representation
- Parametric Cubic Curves
- Parametric Bi-Cubic Surfaces

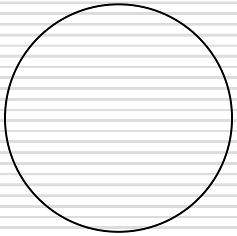
The Utah Teapot



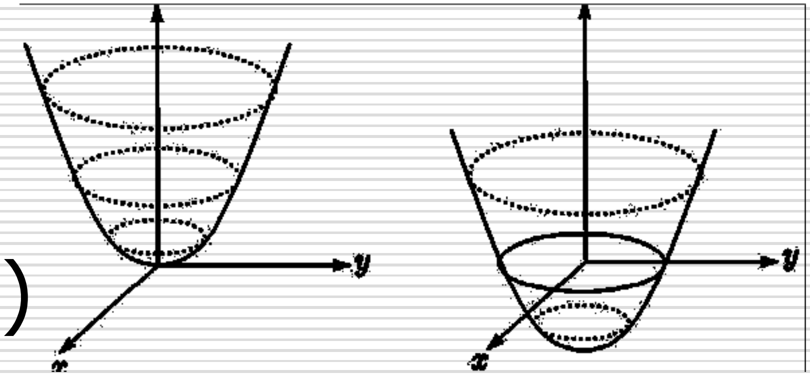
http://en.wikipedia.org/wiki/Utah_teapot
http://www.sjbaker.org/teapot/

Mathematical Curve Representation

- Explicit y=f(x)
  - what if the curve is not a function, e.g., a circle?



- Implicit g(x,y)=0



- Parametric (x(u),y(u))
  - For the circle:

Recall: Plane Equation

- $Ax + By + Cz + D = 0$
  - and (A, B, C) means the normal vector
  - so, given points P_1 , P_2, and P_3 on the plane
  - $(A, B, C) = P_1P_2 \times P_1P_3$
  - what happened if $(A, B, C) = (0,0,0)$?
  - the distance from a vertex (x, y, z) to the plane is

$$d = \frac{Ax + By + Cz + D}{\sqrt{A^2 + B^2 + C^2}}$$

Parametric Polynomial Curves

- We will use parametric curves where the functions are all polynomials in the parameter.

$$x(u) = \sum_{k=0}^{n} a_k u^k$$

$$y(u) = \sum_{k=0}^{n} b_k u^k$$

- Advantages:
  - easy (and efficient) to compute
  - infinitely differentiable

Parametric Cubic Curves

- Fix $n = 3$
- The cubic polynomials that define a curve segment $Q(t) = [x(t) \quad y(t) \quad z(t)]^{\mathrm{T}}$ are of the form

$$
\begin{aligned}
x(t) &= a_x t^3 + b_x t^2 + c_x t + d_x, \\
y(t) &= a_y t^3 + b_y t^2 + c_y t + d_y, \\
z(t) &= a_z t^3 + b_z t^2 + c_z t + d_z, \quad 0 \le t \le 1.
\end{aligned}
$$

Parametric Cubic Curves

- The curve segment can be rewrite as

$$Q(t) = [x(t) \quad y(t) \quad z(t)]^{\mathrm{T}} = C \bullet T$$

- where $T = [t^3 \quad t^2 \quad t \quad 1]^{\mathrm{T}}$

$$C = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix}$$

Tangent Vector

$$\frac{d}{dt}Q(t) = Q'(t) = \left[\frac{d}{dt}x(t) \quad \frac{d}{dt}y(t) \quad \frac{d}{dt}z(t) \right]^{\mathrm{T}}$$

$$= \frac{d}{dt}C \bullet T = C \bullet \left[3t^2 \quad 2t \quad 1 \quad 0 \right]^{\mathrm{T}}$$

$$= \left[3a_x t^2 + 2b_x t + c_x \quad 3a_y t^2 + 2b_y t + c_y \quad 3a_z t^2 + 2b_z t + c_z \right]^{\mathrm{T}}$$

Three Types of Parametric Cubic Curves

- Hermite Curves
  - defined by two **endpoints** and two endpoint **tangent vectors**
- Bézier Curves
  - defined by two **endpoints** and two **control points** which control the endpoint' **tangent vectors**
- Splines
  - defined by four **control points**

Parametric Cubic Curves

- $Q(t) = C \bullet T$
- rewrite the coefficient matrix as $C = G \bullet M$
  - where M is a 4x4 **basis matrix**, G is called the **geometry matrix**
  - so

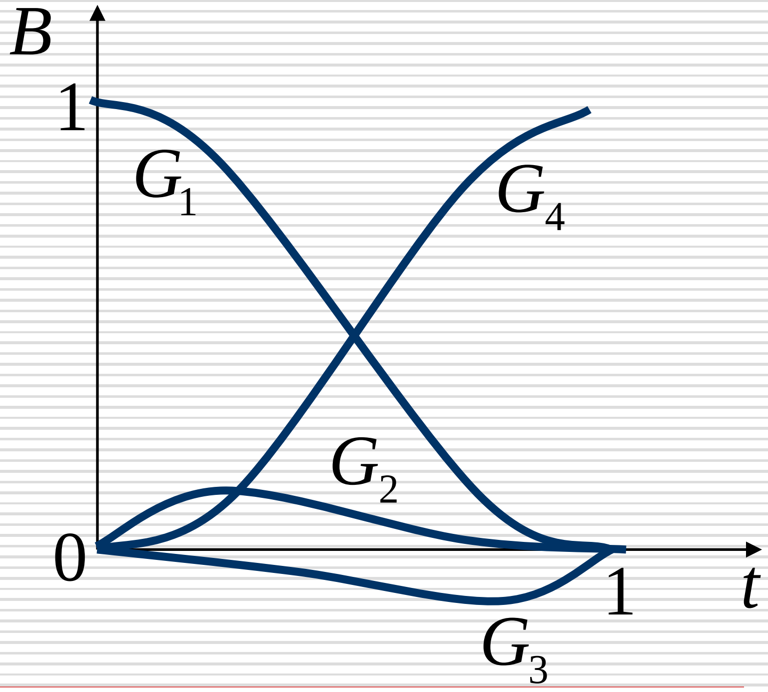
$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} G_1 & G_2 & G_3 & G_4 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

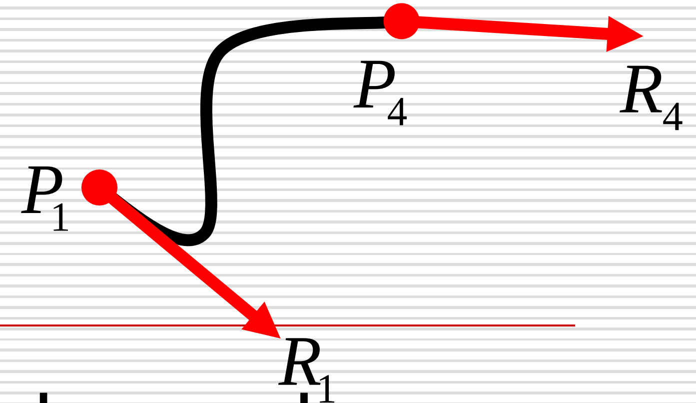
4 endpoints or tangent vectors

Parametric Cubic Curves

- $Q(t) = G \bullet M \bullet T = G \bullet B$

  where $B = M \bullet T$ is called the **blending functions**



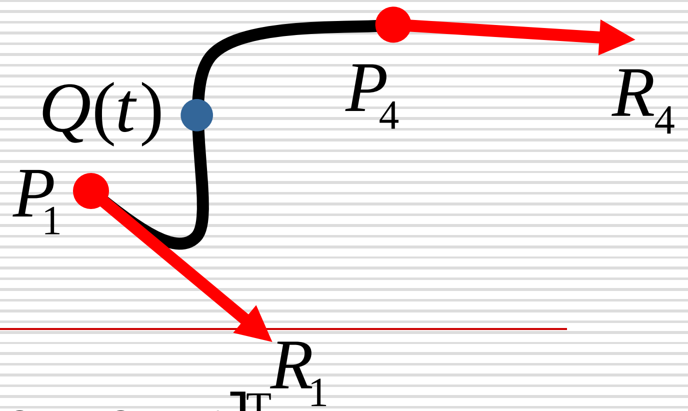


Hermite Curves

- Given the endpoints P_1 and P_4 and tangent vectors at them R_1 and R_4
- What is
  - **Hermite basis matrix** M_{H}
  - **Hermite geometry vector** G_{H}
  - **Hermite blending functions** B_{H}
- by definition

$$G_{\mathrm{H}} = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix}$$

Hermite Curves



- since $Q(0) = P_1 = G_H \bullet M_H \bullet \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$

$$Q(1) = P_4 = G_H \bullet M_H \bullet \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$$Q'(0) = R_1 = G_H \bullet M_H \bullet \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$Q'(1) = R_4 = G_H \bullet M_H \bullet \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix}^T$$

$$G_H = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix} = G_H \bullet M_H \bullet \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

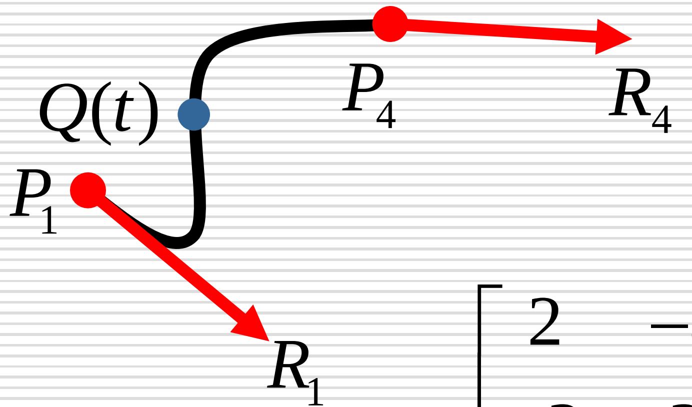
Hermite Curves

- so $$M_H = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$
- and $Q(t) = G_H \bullet M_H \bullet T = G_H \bullet B_H$

$$B_H = \begin{bmatrix} 2t^3 - 3t^2 + 1 & -2t^3 + 3t^2 & t^3 - 2t^2 + t & t^3 - t^2 \end{bmatrix}^T$$

Computing a point

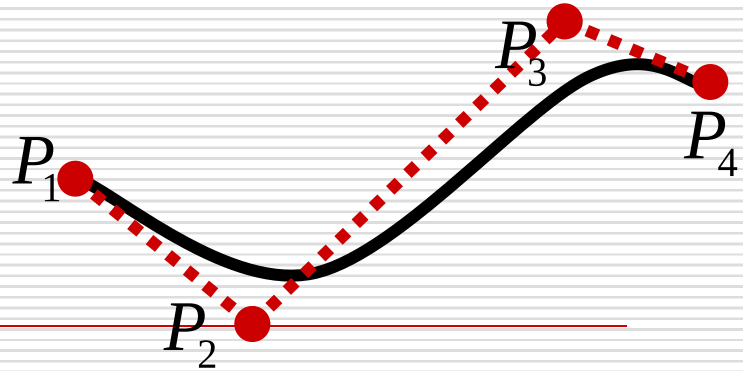
- Given two endpoints P_1 and P_4 and two tangent vectors at them R_1 and R_4



so

$$Q(t) = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Bézier Curves



- Given the endpoints P_1 and P_4 and two control points P_2 and P_3 which determine the endpoints' tangent vectors, such that $R_1 = Q'(0) = 3(P_2 - P_1)$

$$R_4 = Q'(1) = 3(P_4 - P_3)$$

- What is
  - **Bézier basis matrix** M_B
  - **Bézier geometry vector** G_B
  - **Bézier blending functions** B_B

Bézier Curves

- by definition $G_B = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}$
- then $G_H = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix}$

$$= \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} = G_B \bullet M_{HB}$$

- so $Q(t) = G_H \bullet M_H \bullet T = (G_B \bullet M_{HB}) \bullet M_H \bullet T$

$$= G_B \bullet (M_{HB} \bullet M_H) \bullet T = G_B \bullet M_B \bullet T$$

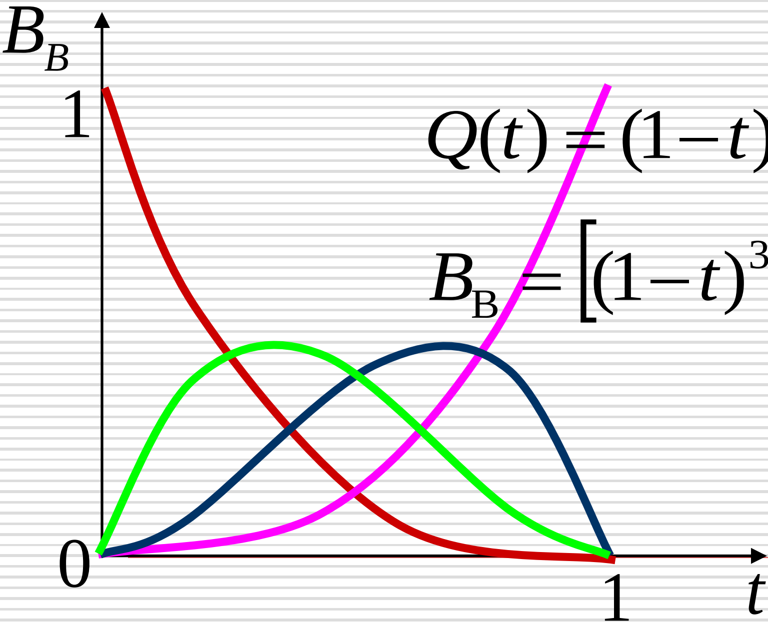
Bézier Curves

- and

$$M_B = M_{HB} \bullet M_H = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

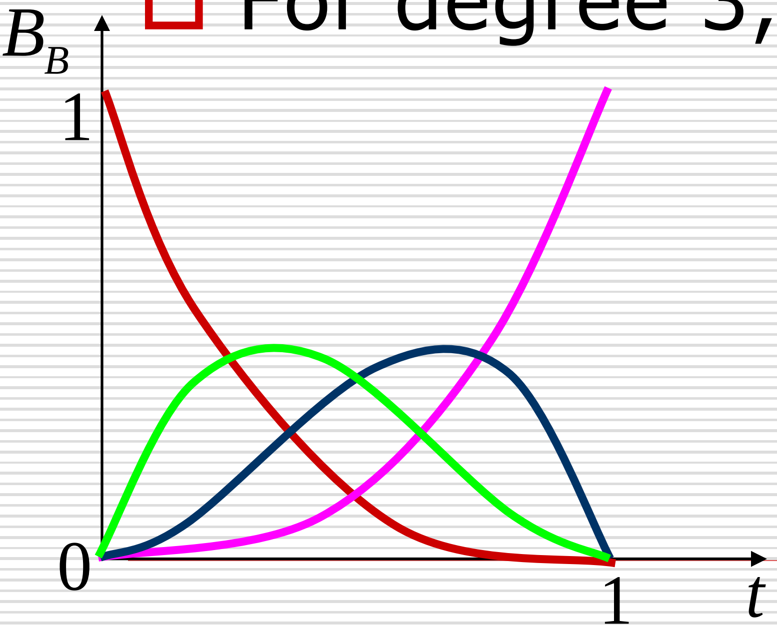
$$B_B = \left[(1-t)^3 \quad 3t(1-t)^2 \quad 3t^2(1-t) \quad t^3 \right]^T$$



Bernstein polynomials

Bernstein Polynomials

- The coefficients of the control points are a set of functions called the **Bernstein polynomials**: $Q(t) = \sum_{i=0}^{n} b_i(t) P_i$
- For degree 3, we have:



$$b_0(t) = (1-t)^3$$

$$b_1(t) = 3t(1-t)^2$$

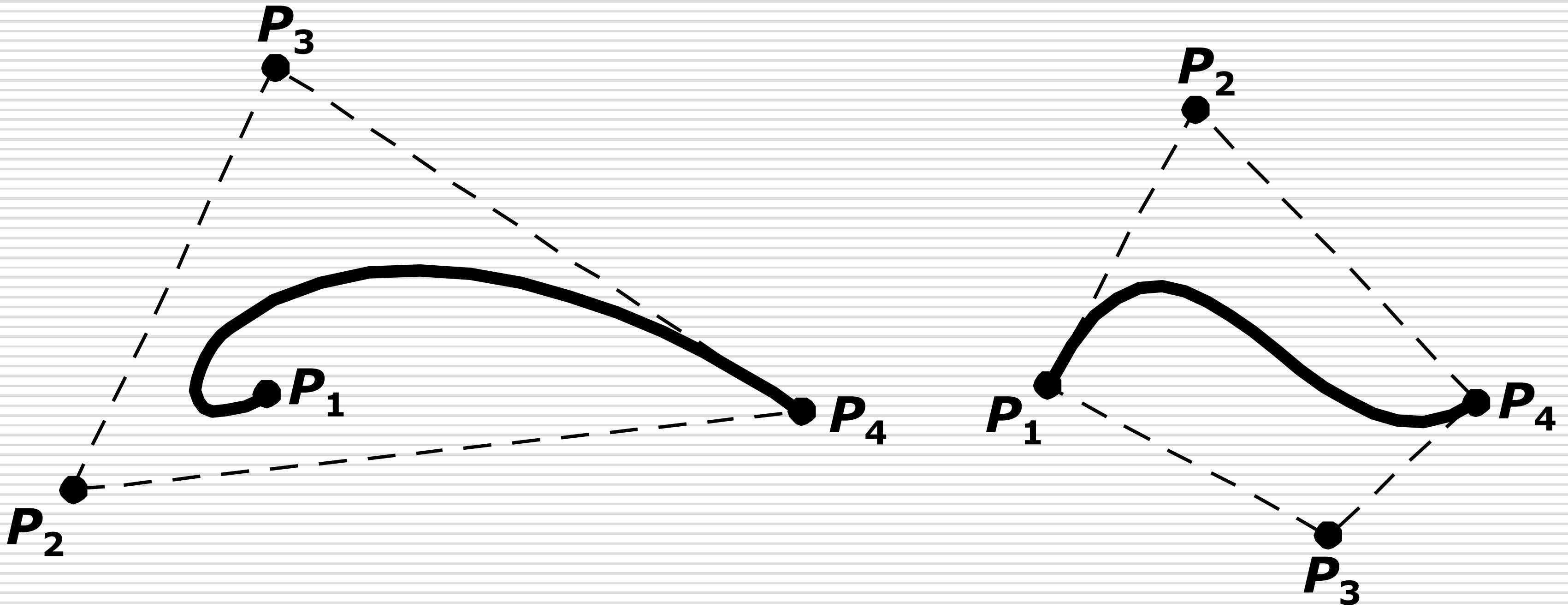
$$b_2(t) = 3t^2(1-t)$$

$$b_3(t) = t^3$$

Bernstein Polynomials

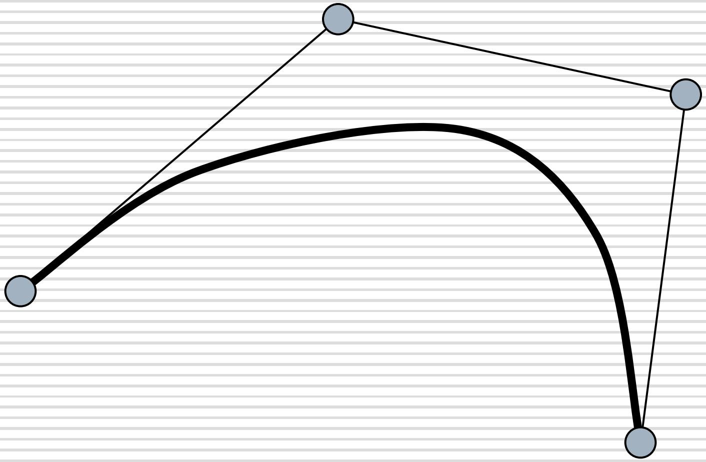
- Useful properties on the interval [0,1]:
  - each is between 0 and 1
  - sum of all four is exact 1
    - a.k.a., a "partition of unity"
- These together imply that the curve lines within the **convex hull** of its control points.

Convex Hull



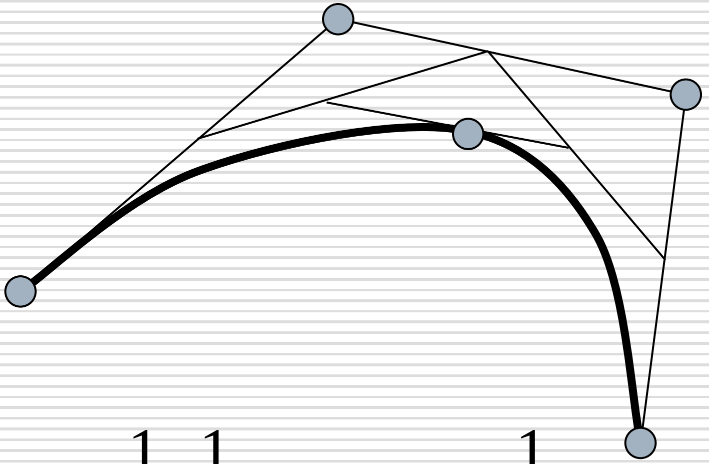
Subdividing Bézier Curves

- $Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$
- How to draw the curve ?
- How to convert it to be line-segments ?



Subdividing Bézier Curves (de Casteljau's algorithm)

- $Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$
- How to draw the curve ?
- How to convert it to be line-segments ?

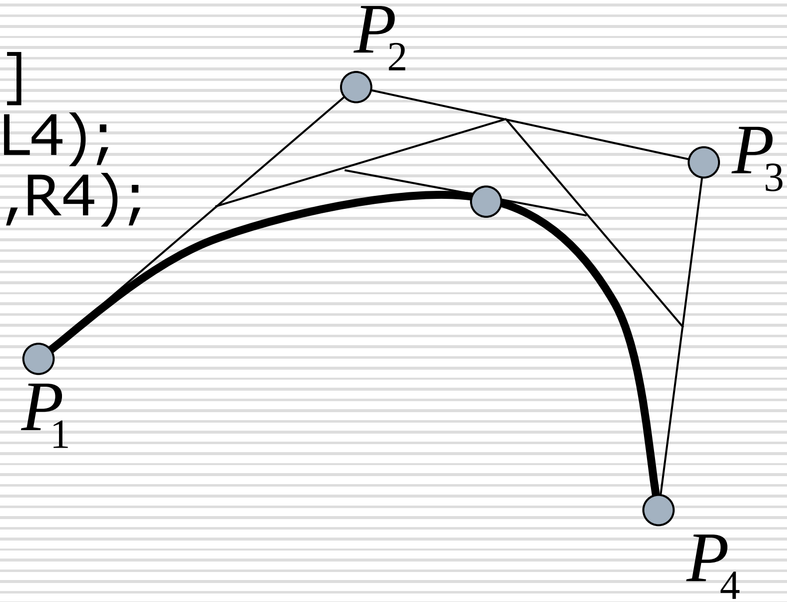


$$Q(\frac{1}{2}) = \frac{1}{8}P_1 + \frac{3}{8}P_2 + \frac{3}{8}P_3 + \frac{1}{8}P_4$$

$$= \frac{1}{2}(\frac{1}{2}(\frac{1}{2}(P_1 + P_2) + \frac{1}{2}(P_2 + P_3)) + \frac{1}{2}(\frac{1}{2}(P_3 + P_4) + \frac{1}{2}(P_2 + P_3)))$$

Display Bézier Curves

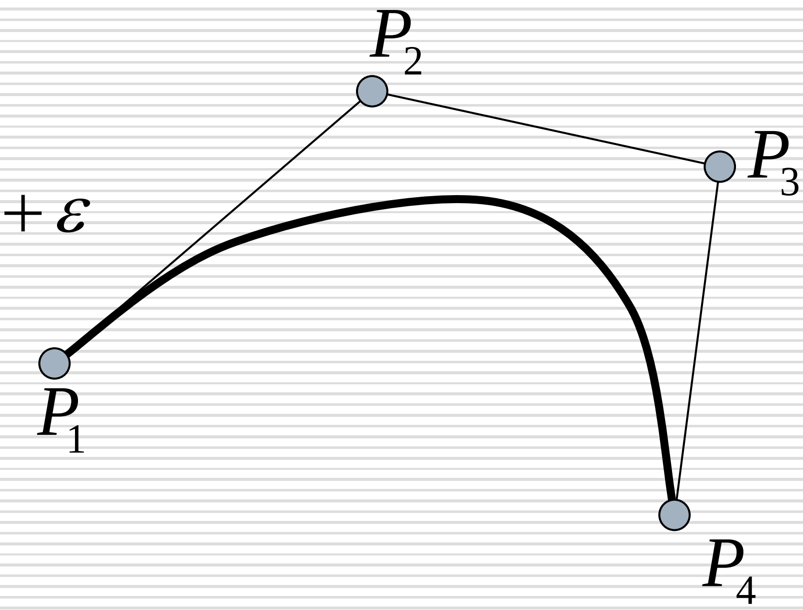
```
DisplayBezier(P1,P2,P3,P4)
begin
    if (FlatEnough(P1,P2,P3,P4))
        Line(P1,P4);
    else
        Subdivide(P[])=>L[],R[]
        DisplayBezier(L1,L2,L3,L4);
        DisplayBezier(R1,R2,R3,R4);
end;
```



Testing for Flatness

- Compare total length of control polygon to length of line connecting endpoints

$$\frac{|P_1 - P_2| + |P_2 - P_3| + |P_3 - P_4|}{|P_1 - P_4|} < 1 + \varepsilon$$

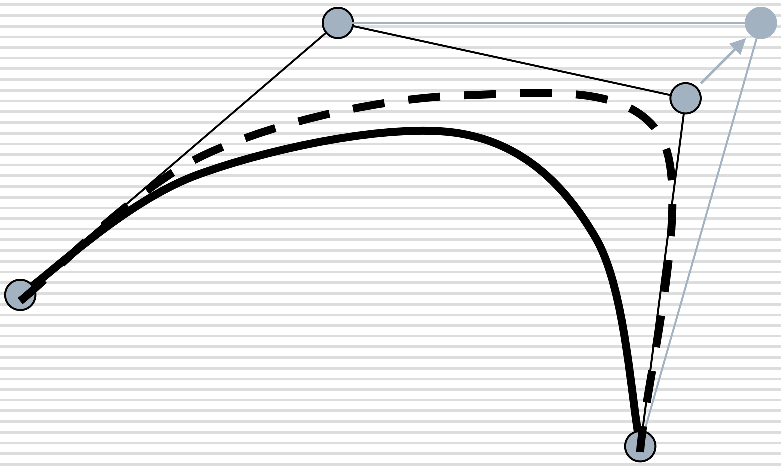


What do we want for a curve?

- Local control
- Interpolation
- Continuity

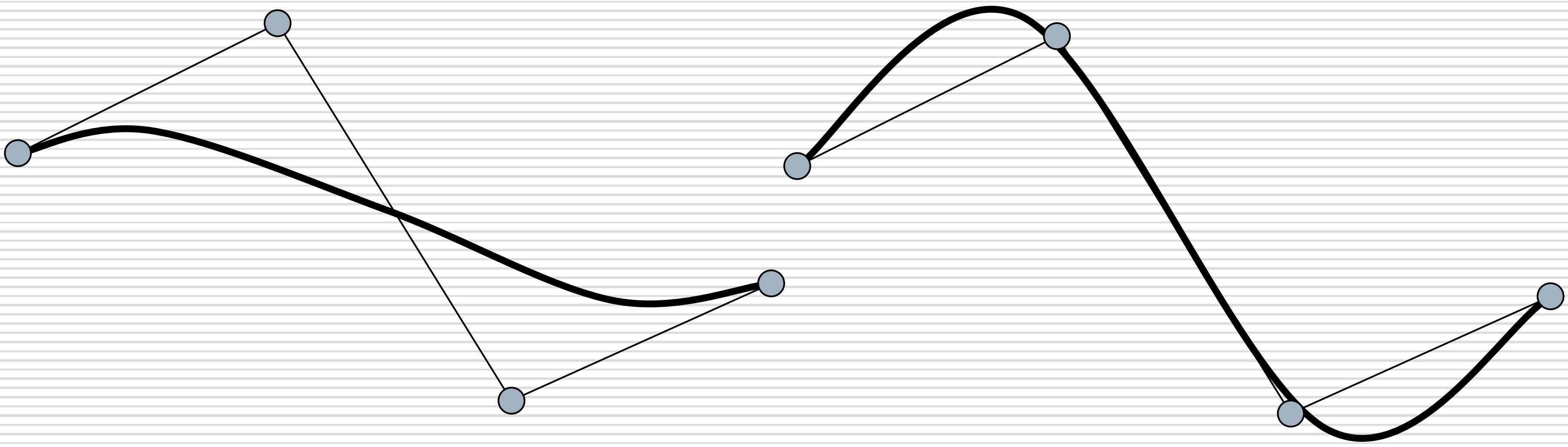
Local Control

- One problem with Bézier curve is that every control points affect every point on the curve (except for endpoints). Moving a single control point affects the whole curve.
- We'd like to have local control, that is, have each control point affect some well-defined neighborhood around that point.

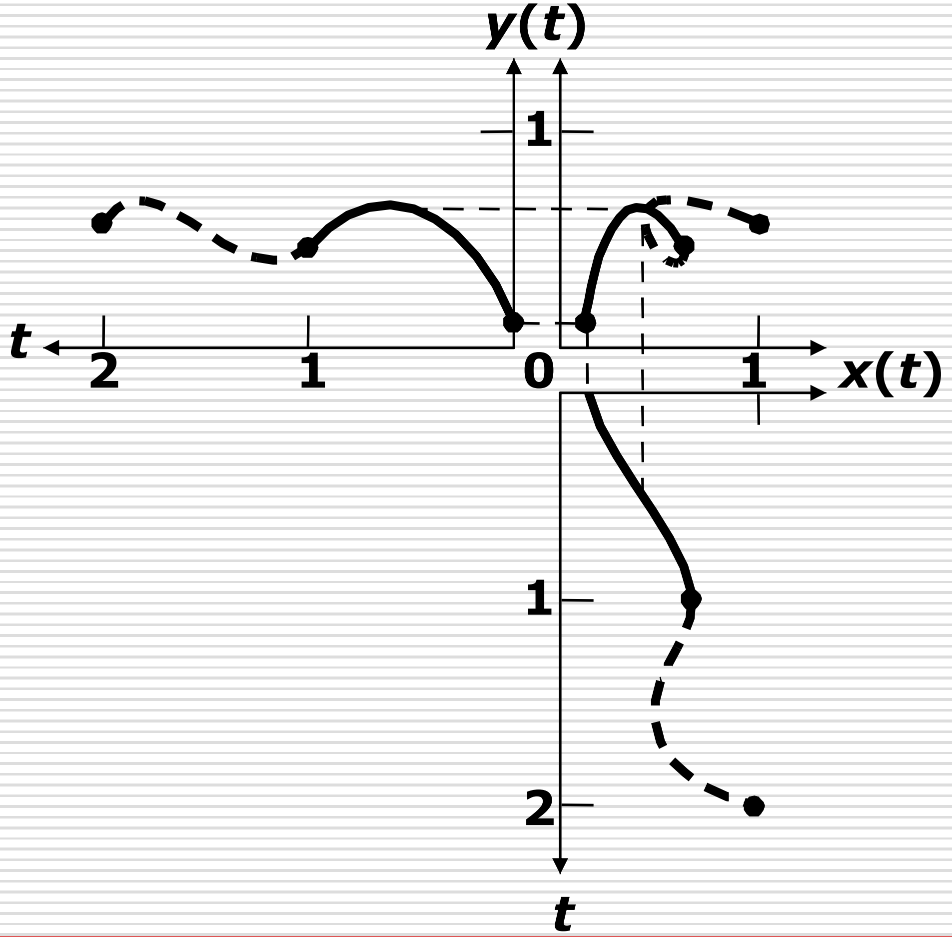


Interpolation

- Bézier curves are approximating. The curve does not necessarily pass through all the control points. We'd like to have a curve that is interpolating, that is, that always passes through every control points.



Continuity between Curve Segments



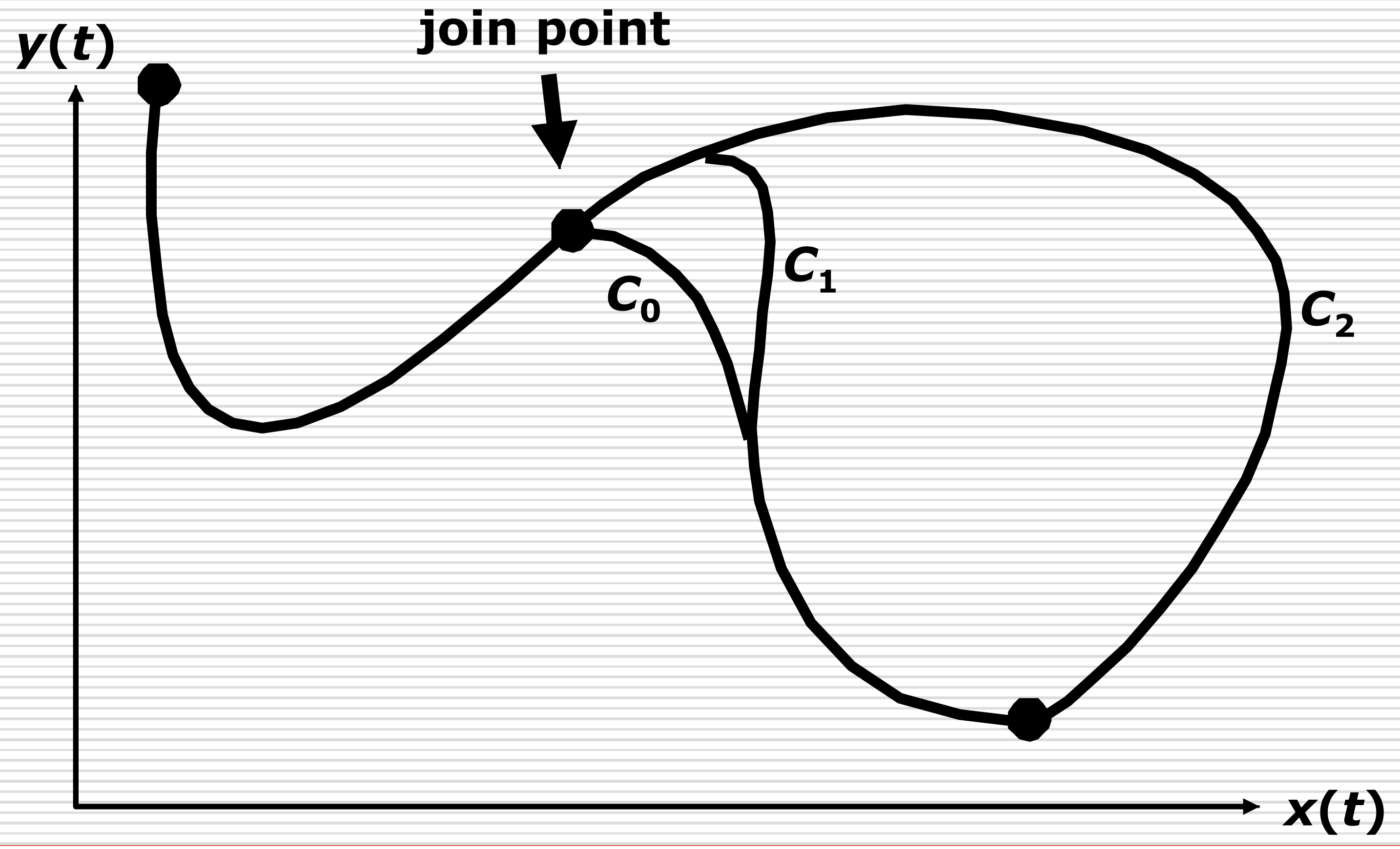
Continuity between Curve Segments

- G^0 geometric continuity
  - two curve segments join together
- G^1 geometric continuity
  - the directions *(but not necessarily the magnitudes)* of the two segments' tangent vectors are equal at a join point

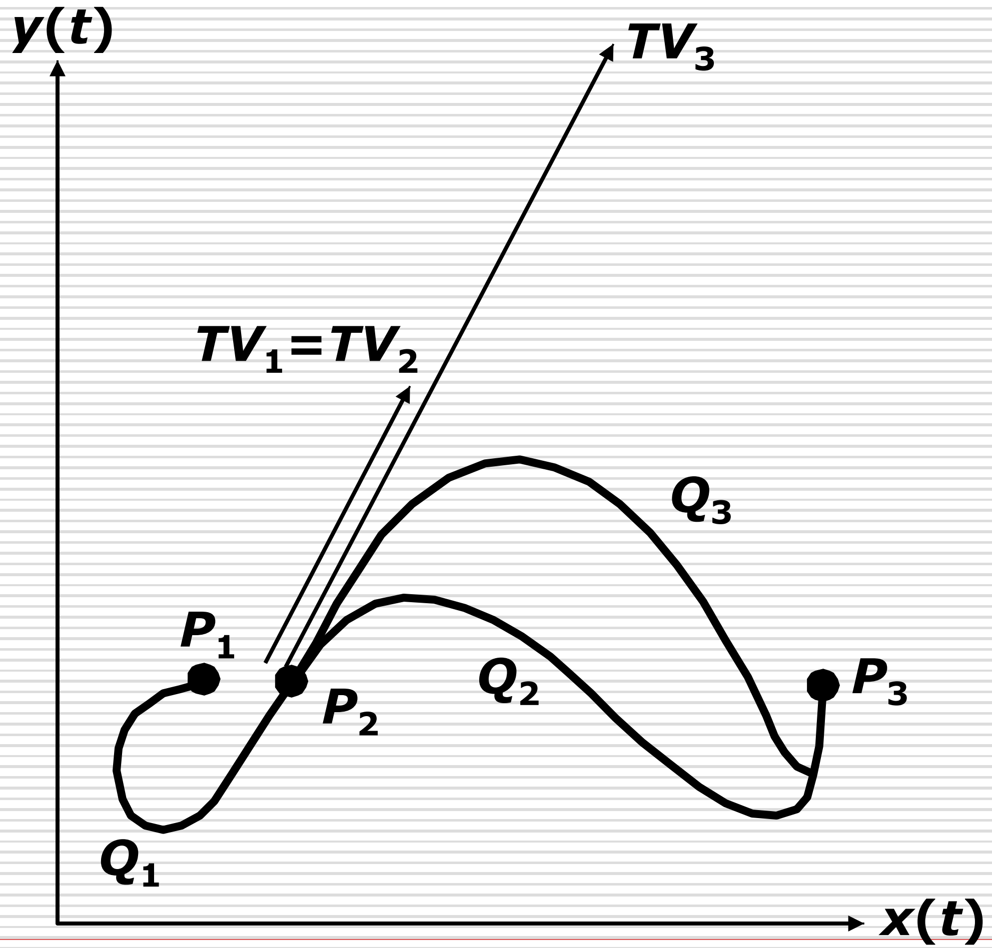
Continuity between Curve Segments

- C^1 continuous
  - the tangent vectors of the two cubic curve segments are equal *(both directions and magnitudes)* at the segments' join point
- C^n continuous
  - the direction and magnitude of $d^n / dt^n [Q(t)]$ through the *n*th derivative are equal at the join point

Continuity between Curve Segments



Continuity between Curve Segments



Bézier Curves → Splines

- Bézier curves have C-infinity continuity on their interiors, but we saw that they do not exhibit local control or interpolate their control points.
- It is possible to define points that we want to interpolate, and then solve for the Bézier control points that will do the job.
- But, you will need as many control points as interpolated points -> high order polynomials -> wiggly curves. (And you still won't have local control.)

Bézier Curves → Splines

- We will splice together a curve from individual Bézier segments. We call these curves **splines**.
- When splicing Bézier together, we need to worry about continuity.

Ensuring C^0 continuity

- Suppose we have a cubic Bézier defined by (V_1, V_2, V_3, V_4), and we want to attach another curve (W_1, W_2, W_3, W_4) to it, so that there is C^0 continuity at the joint.

$$C^0 : Q_V(1) = Q_W(0)$$

- What constraint(s) does this place on (W_1, W_2, W_3, W_4) ?

$$Q_V(1) = Q_W(0) \Rightarrow V_4 = W_1$$

Ensuring C¹ continuity

- Suppose we have a cubic Bézier defined by (V_1, V_2, V_3, V_4), and we want to attach another curve (W_1, W_2, W_3, W_4) to it, so that there is C¹ continuity at the joint.

$$C^0 : Q_V(1) = Q_W(0)$$

$$C^1 : Q'_V(1) = Q'_W(0)$$

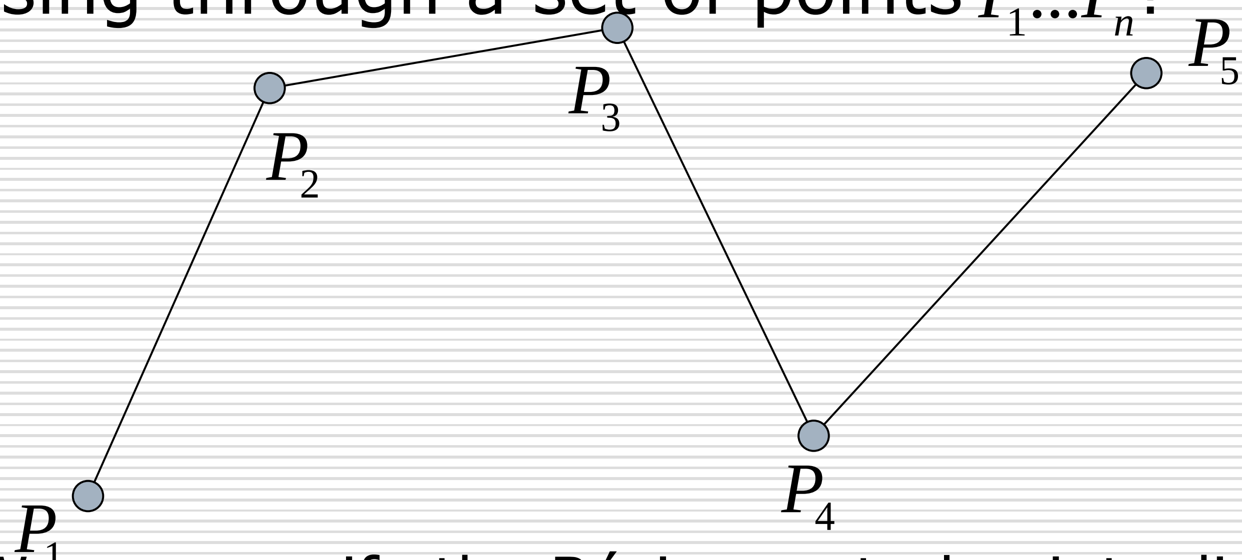
- What constraint(s) does this place on (W_1, W_2, W_3, W_4) ?

$$Q_V(1) = Q_W(0) \Rightarrow V_4 = W_1$$

$$Q'_V(1) = Q'_W(0) \Rightarrow V_4 - V_3 = W_2 - W_1$$

The C^1 Bézier Spline

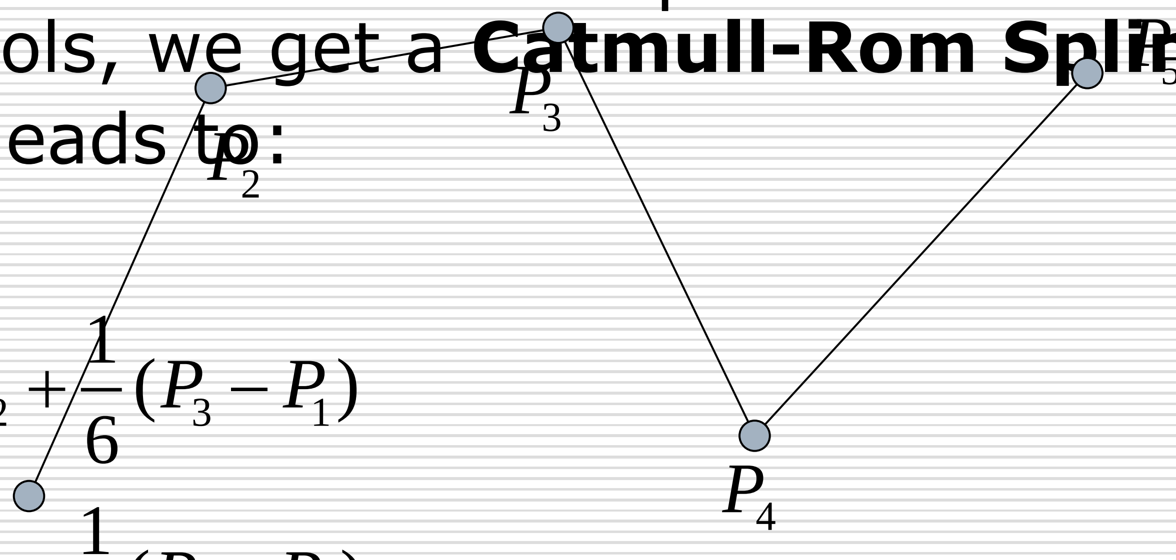
- How then could we construct a curve passing through a set of points $P_1 \dots P_n$?



- We can specify the Bézier control points directly, or we can devise a scheme for placing them automatically…

Catmull-Rom Spline

- If we set each derivative to be one half of the vector between the previous and next controls, we get a **Catmull-Rom Spline**.
- This leads to:



$$V_1 = P_2$$

$$V_2 = P_2 + \frac{1}{6}(P_3 - P_1)$$

$$V_3 = P_3 - \frac{1}{6}(P_4 - P_2)$$

$$V_4 = P_3$$

Catmull-Rom Basis Matrix

$$Q(t) = G_B \bullet M_B \bullet T$$

$$= G_B \bullet \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet T \quad G_B = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{6} & 1 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 1 & \frac{-1}{6} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

$$Q(t) = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Ensuring C^2 continuity

- Suppose we have a cubic Bézier defined by (V_1, V_2, V_3, V_4), and we want to attach another curve (W_1, W_2, W_3, W_4) to it, so that there is C^2 continuity at the joint.

$$Q_V(1) = Q_W(0) \Rightarrow V_4 = W_1$$

$$Q'_V(1) = Q'_W(0) \Rightarrow V_4 - V_3 = W_2 - W_1$$

$$Q''_V(1) = Q''_W(0) \Rightarrow V_2 - 2V_3 + V_4 = W_1 - 2W_2 + W_3$$

$$\Downarrow$$

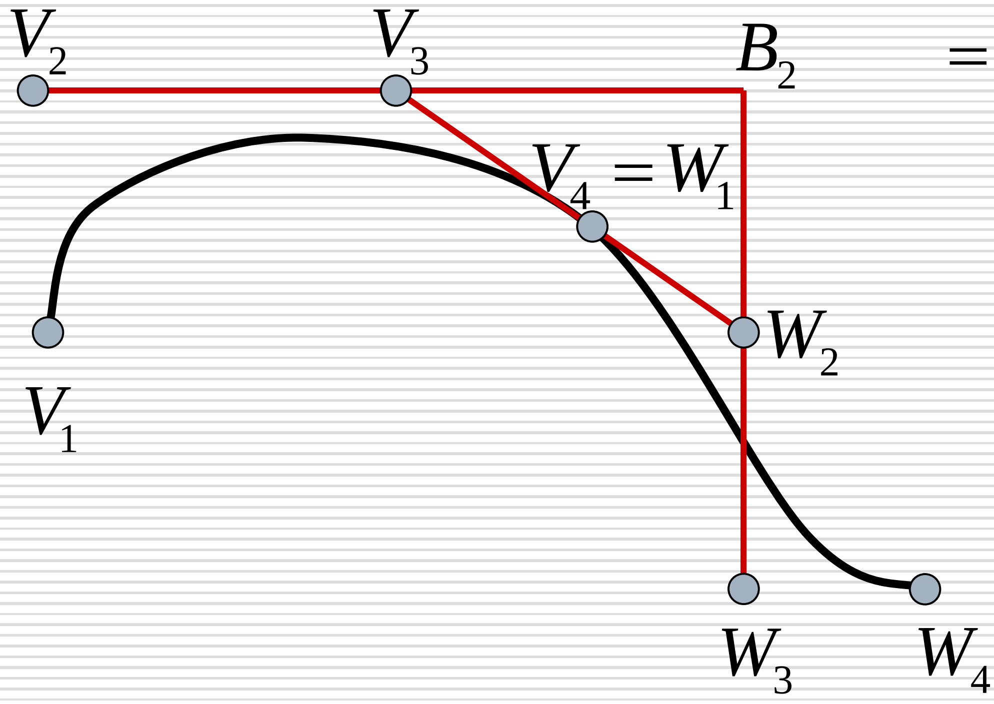
$$W_3 = V_2 - 4V_3 + 4V_4$$

B-Spline

- Instead of specifying the Bézier control points themselves, let's specify the corners of the A-frames in order to build a C^2 continuous spline.

B-Spline

$$W_3 = V_2 - 4V_3 + 4V_4$$
$$= 2(2V_4 - V_3) - (2V_3 - V_2)$$
$$= 2W_2 - B_2$$



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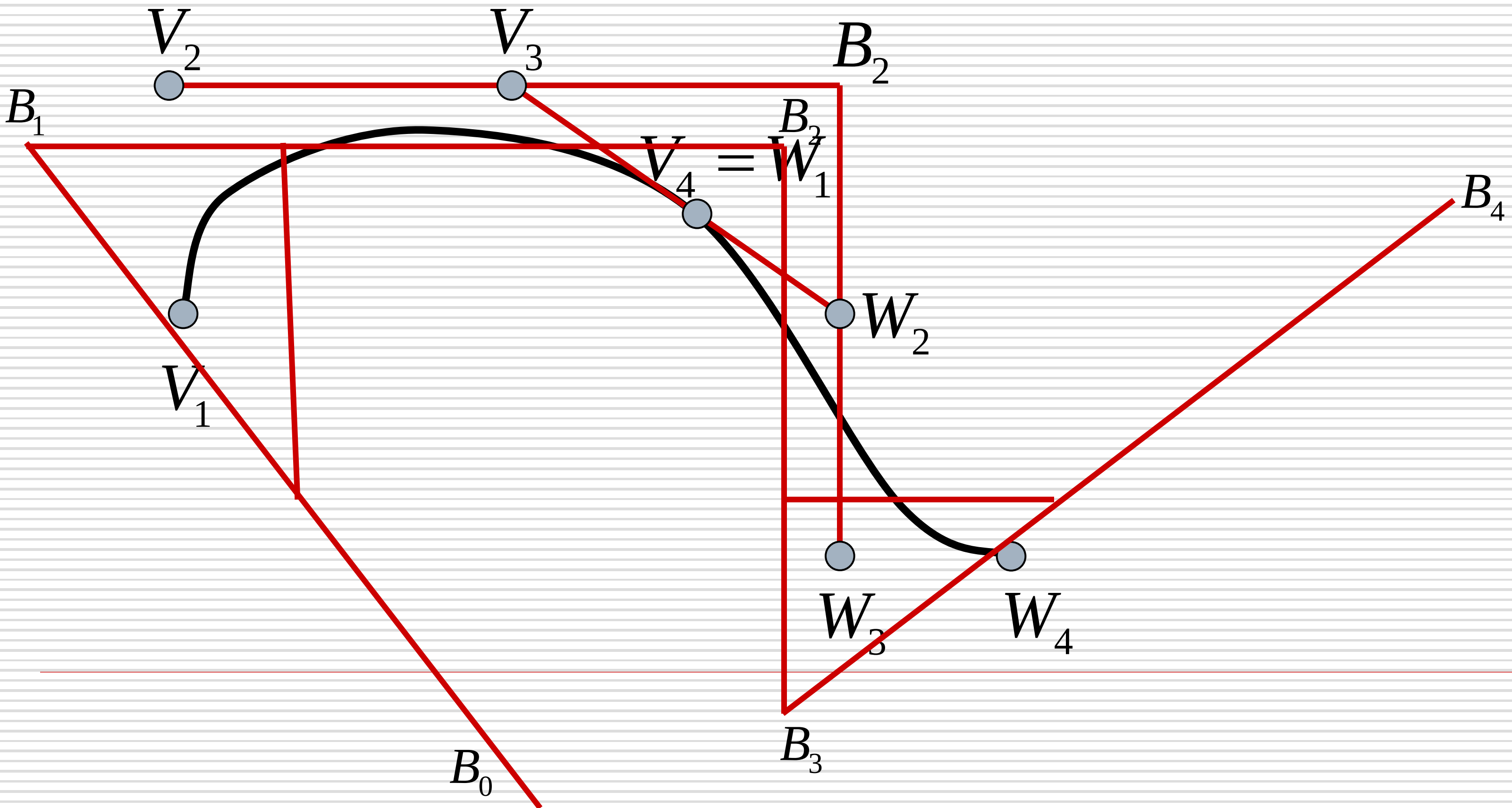

B-Spline

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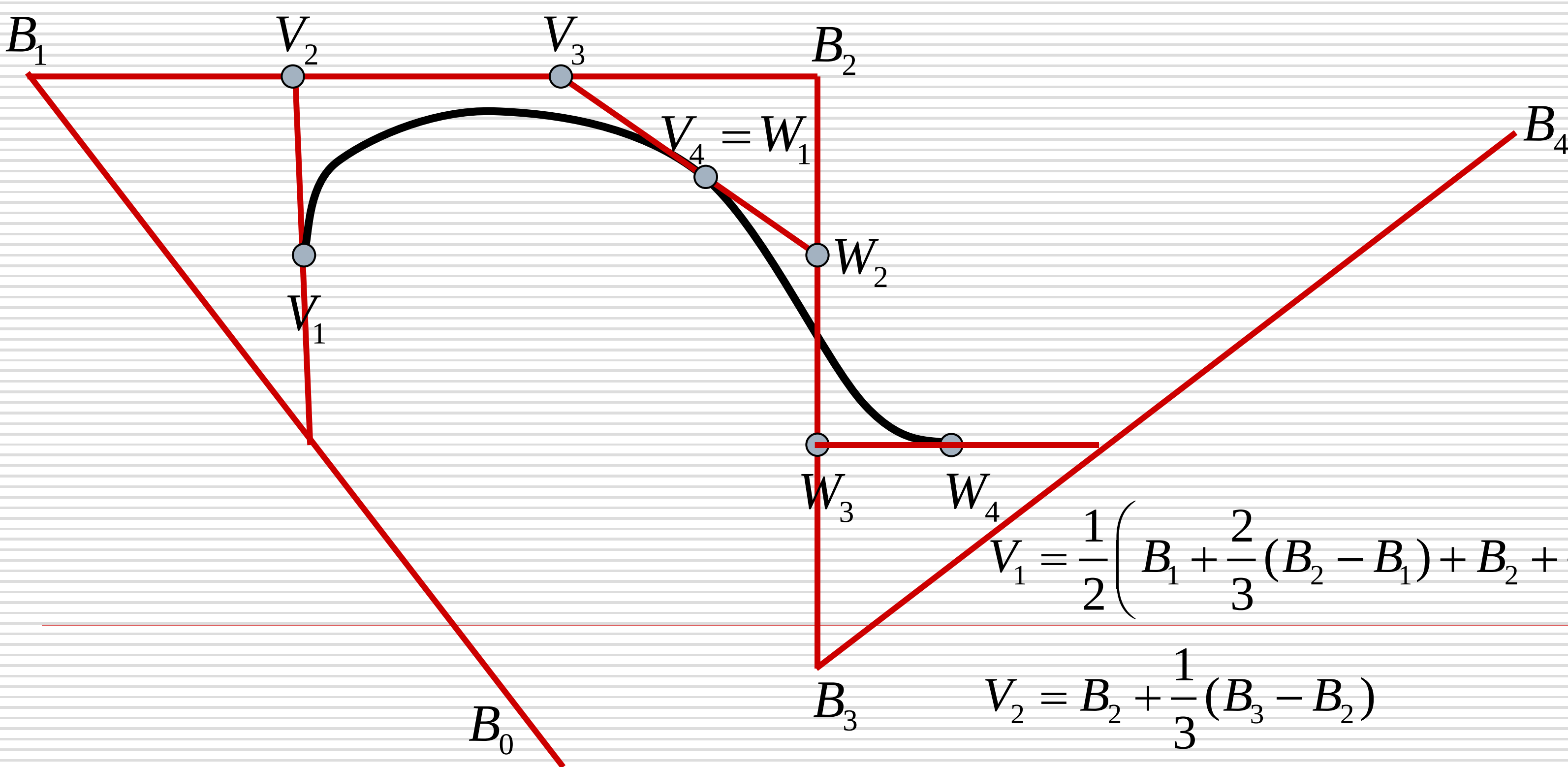
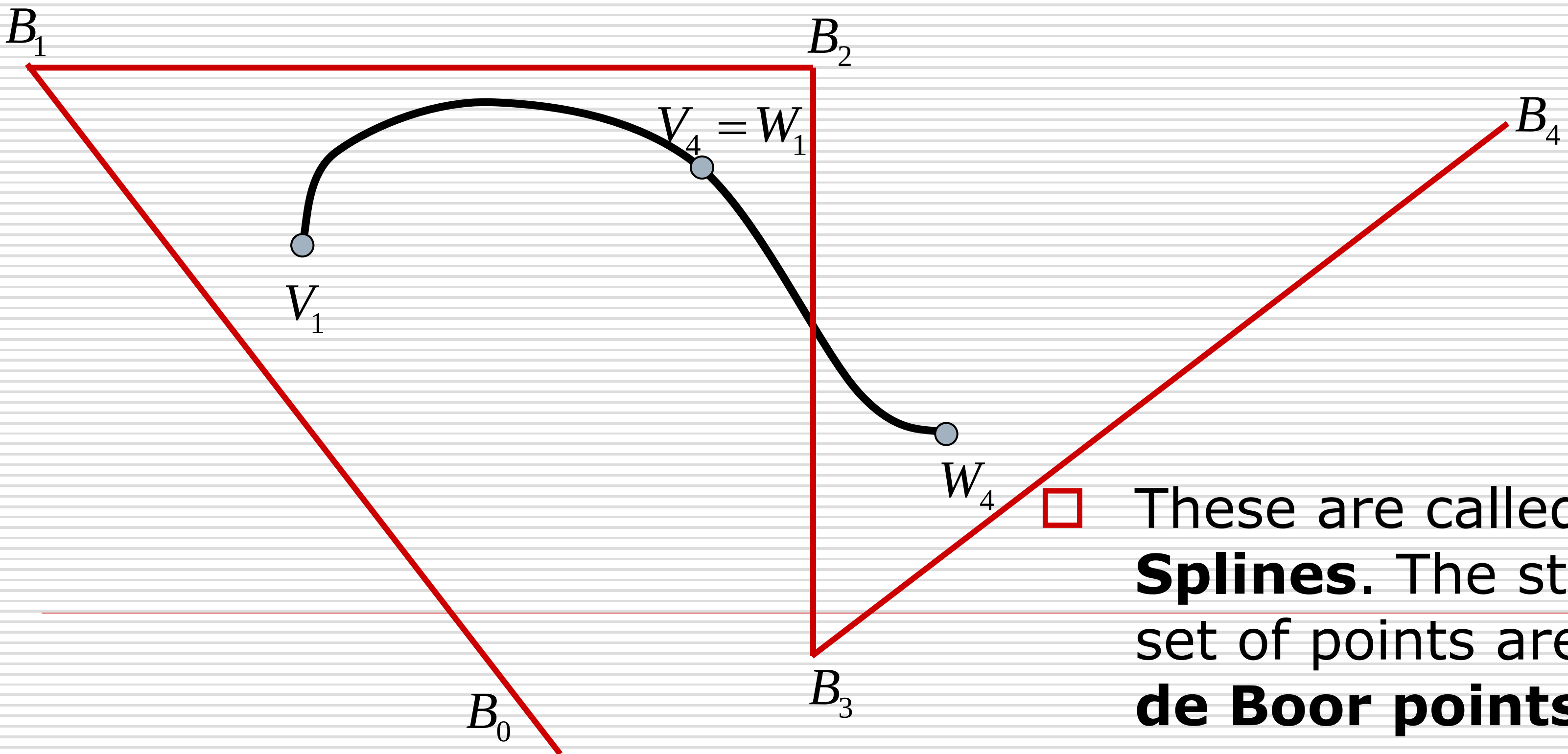


$$V_1 = \frac{1}{2}\left(B_1 + \frac{2}{3}(B_2 - B_1) + B_2 + \frac{1}{3}(B_3 - B_2) \right)$$

$$V_2 = B_2 + \frac{1}{3}(B_3 - B_2)$$

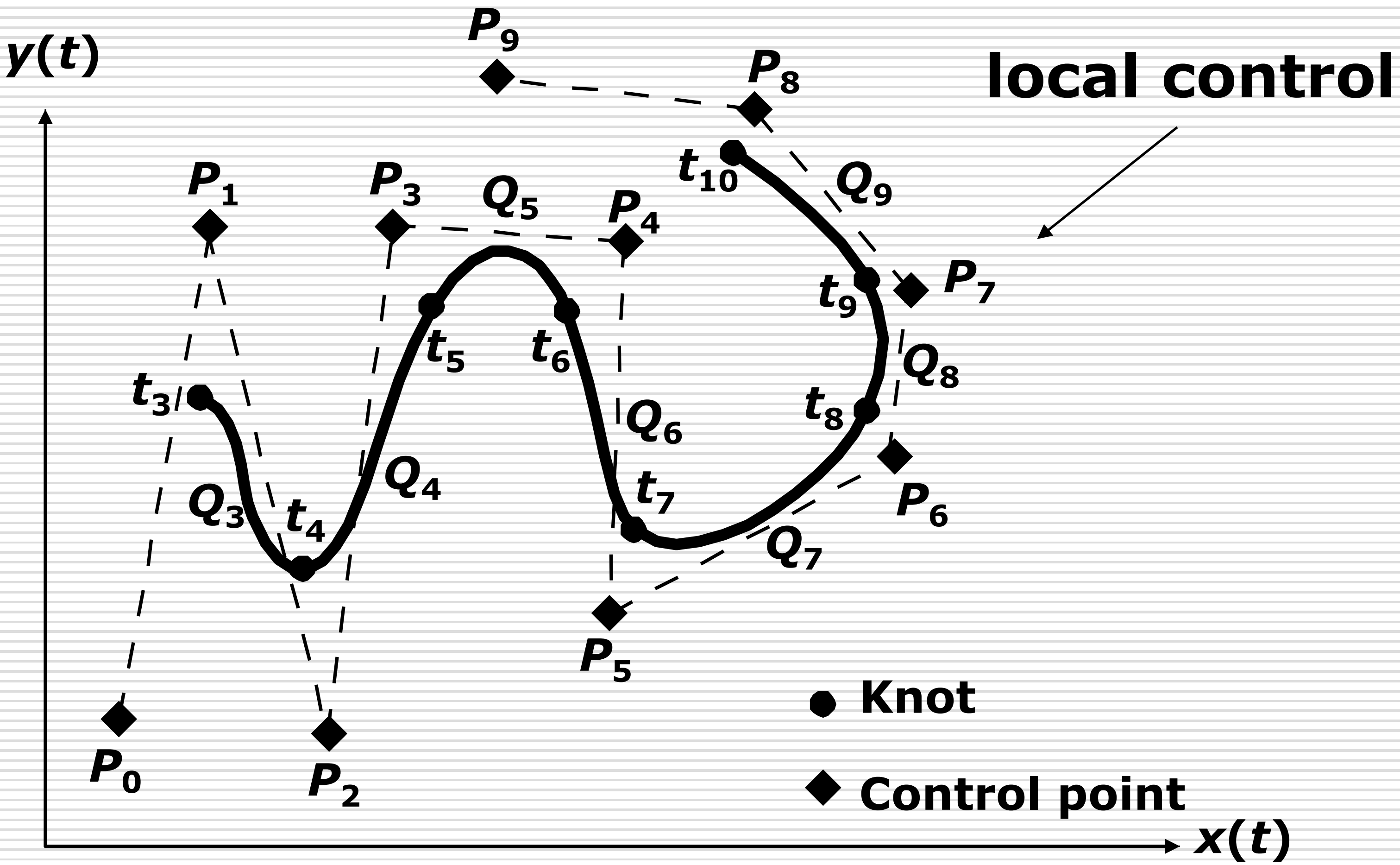
B-Spline

- Instead of specifying the Bézier control points themselves, let's specify the corners of the A-frames in order to build a C^2 continuous spline.



- These are called **B-Splines**. The starting set of points are called **de Boor points**.

B-Spline



P_9
P_8
local control
$y(t)$
t_{10}
Q_9
P_1
P_3
Q_5
P_4
P_7
t_9
t_5
t_6
Q_8
t_3
t_8
Q_6
Q_4
P_6
t_7
Q_3
t_4
Q_7
P_5
Knot
P_0
P_2
Control point
$x(t)$

Uniform NonRational B-Splines

- cubic B-Spline
  - has $m+1$ control points $P_0, P_1, \ldots, P_m, m \geq 3$
  - has $m-2$ cubic polynomial curve segments $Q_3, Q_4, \ldots, Q_m$
- uniform
  - the knots are spaced at equal intervals of the parameter t
- non-rational
  - not rational cubic polynomial curves

Uniform NonRational B-Splines

- curve segment Q_i is defined by points $P_{i-3}, P_{i-2}, P_{i-1}, P_i$, thus
- **B-Spline geometry matrix**

$$G_{\mathrm{Bs}_i} = \begin{bmatrix} P_{i-3} & P_{i-2} & P_{i-1} & P_i \end{bmatrix}, \quad 3 \le i \le m$$

- if $T_i = \begin{bmatrix} (t-t_i)^3 & (t-t_i)^2 & (t-t_i) & 1 \end{bmatrix}^{\mathrm{T}}$
- then $Q_i(t) = G_{\mathrm{Bs}_i} \bullet M_{\mathrm{Bs}} \bullet T_i, \quad t_i \le t \le t_{i+1}$

Uniform NonRational B-Splines

- so **B-Spline basis matrix**

$$M_{\mathrm{Bs}} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- **B-Spline blending functions**

$$B_{\mathrm{Bs}} = \frac{1}{6}\left[(1-\mathrm{t})^3 \quad 3t^3 - 6t^2 + 4 \quad -3t^3 + 3t^2 + 3t + 1 \quad t^3\right]^{\mathrm{T}}, \quad 0 \le t \le 1$$

NonUniform NonRational B-Splines

- the **knot-value sequence** is a nondecreasing sequence
- allow **multiple knot** and the number of identical parameter is the **multiplicity**
  - Ex. (0,0,0,0,1,1,2,3,4,4,5,5,5,5)
- so

$$Q_i(t) = P_{i-3} \bullet B_{i-3,4}(t) + P_{i-2} \bullet B_{i-2,4}(t) + P_{i-1} \bullet B_{i-1,4}(t) + P_i \bullet B_{i,4}(t)$$

NonUniform NonRational B-Splines

- where $B_{i,j}(t)$ is jth-order blending function for weighting control point P_i

$$B_{i,1}(t) = \begin{cases} 1, & t_i \le t \le t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{i,2}(t) = \frac{t - t_i}{t_{i+1} - t_i} B_{i,1}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B_{i+1,1}(t)$$

$$B_{i,3}(t) = \frac{t - t_i}{t_{i+2} - t_i} B_{i,2}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} B_{i+1,2}(t)$$

$$B_{i,4}(t) = \frac{t - t_i}{t_{i+3} - t_i} B_{i,3}(t) + \frac{t_{i+4} - t}{t_{i+4} - t_{i+1}} B_{i+1,3}(t)$$

Knot Multiplicity & Continuity

- since $Q(t_i)$ is within the convex hull of P_{i-3}, P_{i-2}, and P_{i-1}
- if $t_i = t_{i+1}$, $Q(t_i)$ is within the convex hull of P_{i-3}, P_{i-2}, and P_{i-1} and the convex hull of P_{i-2}, P_{i-1}, and P_i, so it will lie on $\overline{P_{i-2}P_{i-1}}$
- if $t_i = t_{i+1} = t_{i+2}$, $Q(t_i)$ will lie on P_{i-1}
- if $t_i = t_{i+1} = t_{i+2} = t_{i+3}$, $Q(t_i)$ will lie on both P_{i-1} and P_i, and the curve becomes broken

Knot Multiplicity & Continuity

- multiplicity 1 : C^2 continuity
- multiplicity 2 : C^1 continuity
- multiplicity 3 : C^0 continuity
- multiplicity 4 : no continuity

NURBS:
NonUniform Rational B-Splines

- rational
  - $x(t)$, $y(t)$, and $z(t)$ are defined as the ratio of two cubic polynomials
- rational cubic polynomial curve segments are ratios of polynomials

$$x(t) = \frac{X(t)}{W(t)} \quad y(t) = \frac{Y(t)}{W(t)} \quad z(t) = \frac{Z(t)}{W(t)}$$

- can be Bézier, Hermite, or B-Splines

Parametric Bi-Cubic Surfaces

- parametric cubic curves are $Q(t) = G \bullet M \bullet T$
- so, parametric bi-cubic surfaces are

  $$Q(s) = G \bullet M \bullet S$$

- if we allow the points in G to vary in 3D along some path, then

  $$Q(s,t) = \begin{bmatrix} G_1(t) & G_2(t) & G_3(t) & G_4(t) \end{bmatrix} \bullet M \bullet S$$

- since $G_i(t)$ are cubics

  $$G_i(t) = \boldsymbol{G}_i \bullet M \bullet T, \text{ where } \boldsymbol{G}_i = \begin{bmatrix} \boldsymbol{g}_{i1} & \boldsymbol{g}_{i2} & \boldsymbol{g}_{i3} & \boldsymbol{g}_{i4} \end{bmatrix}$$

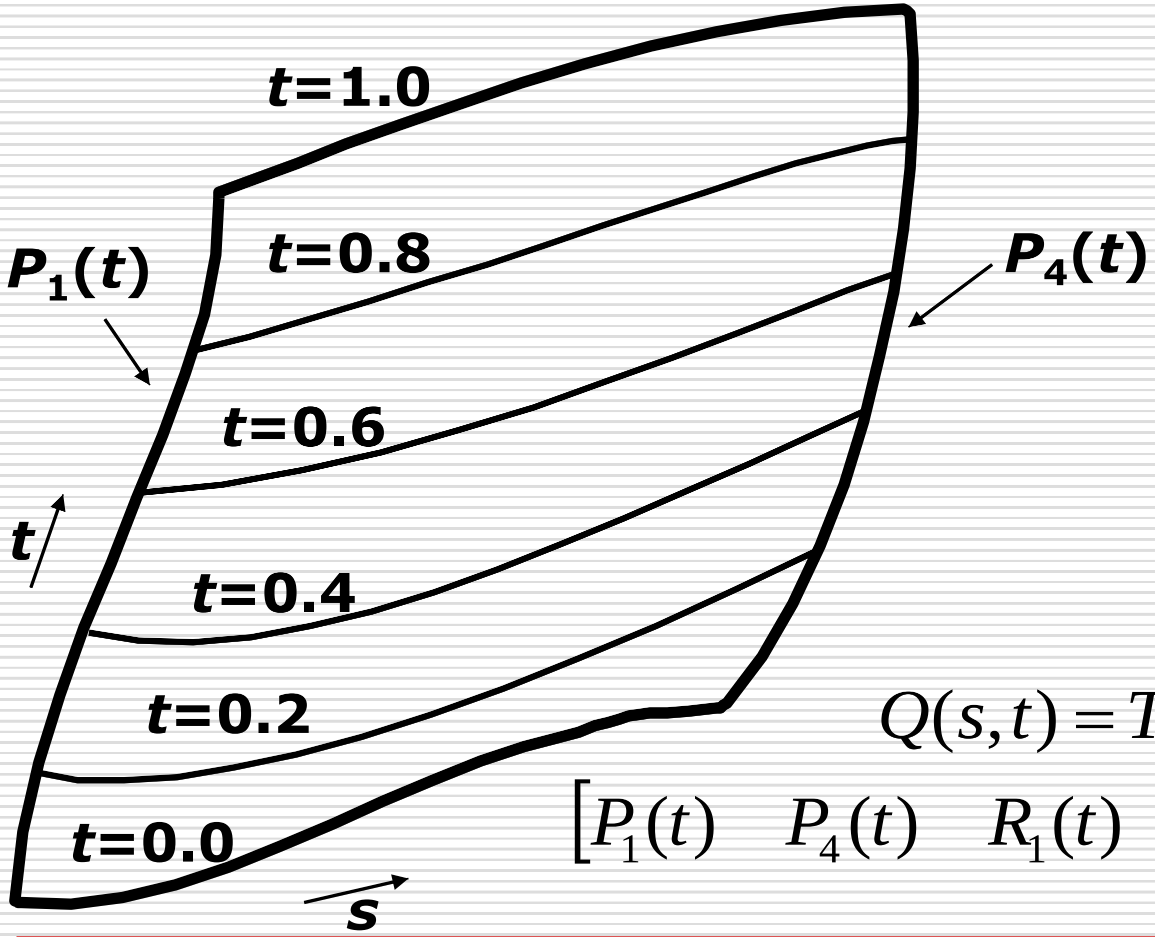
Parametric Bi-Cubic Surfaces

- so

$$Q(s,t) = T^{\mathrm{T}} \bullet M^{\mathrm{T}} \bullet \begin{bmatrix} \boldsymbol{g}_{11} & \boldsymbol{g}_{21} & \boldsymbol{g}_{31} & \boldsymbol{g}_{41} \\ \boldsymbol{g}_{12} & \boldsymbol{g}_{22} & \boldsymbol{g}_{32} & \boldsymbol{g}_{42} \\ \boldsymbol{g}_{13} & \boldsymbol{g}_{23} & \boldsymbol{g}_{33} & \boldsymbol{g}_{43} \\ \boldsymbol{g}_{14} & \boldsymbol{g}_{24} & \boldsymbol{g}_{34} & \boldsymbol{g}_{44} \end{bmatrix} \bullet M \bullet S$$

$$= T^{\mathrm{T}} \bullet M^{\mathrm{T}} \bullet \boldsymbol{G} \bullet M \bullet S, \qquad 0 \le s,t \le 1$$

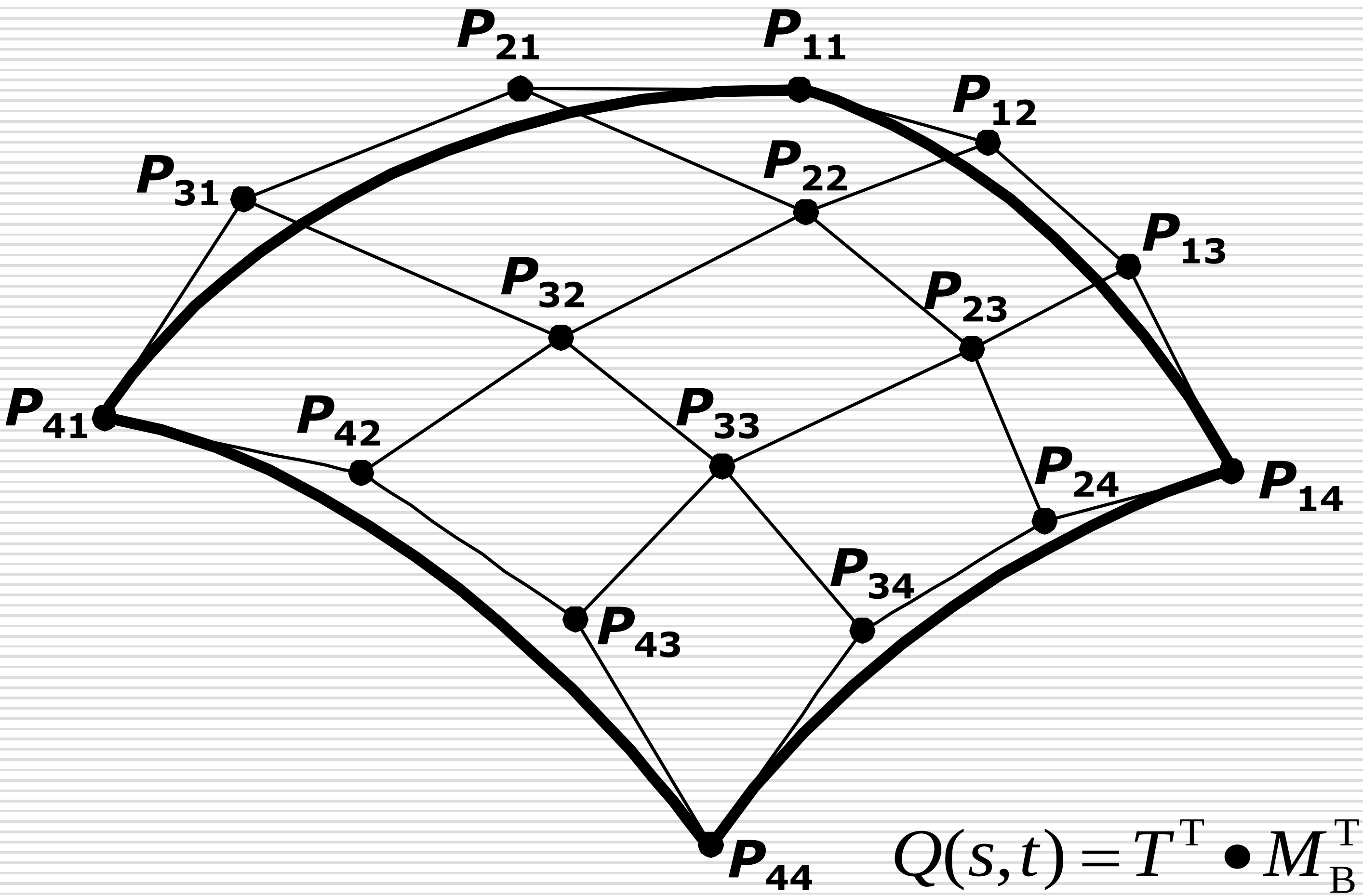
Hermite Surfaces



$$Q(s,t) = T^{\mathrm{T}} \bullet M_{\mathrm{H}}^{\mathrm{T}} \bullet \boldsymbol{G}_{\mathrm{H}} \bullet M_{\mathrm{H}} \bullet S$$

$$\begin{bmatrix} P_1(t) & P_4(t) & R_1(t) & R_4(t) \end{bmatrix} = \boldsymbol{G}_{\mathrm{H}} \bullet M_{\mathrm{H}} \bullet T$$

Bézier Surfaces



$$Q(s,t) = T^{\mathrm{T}} \bullet M_{\mathrm{B}}^{\mathrm{T}} \bullet \boldsymbol{G}_{\mathrm{B}} \bullet M_{\mathrm{B}} \bullet S$$

Normals to Surfaces

$$\frac{\partial}{\partial s}Q(s,t) = T^{\mathrm{T}} \bullet M^{\mathrm{T}} \bullet \boldsymbol{G} \bullet M \bullet \frac{\partial}{\partial s}S$$

$$= T^{\mathrm{T}} \bullet M^{\mathrm{T}} \bullet \boldsymbol{G} \bullet M \bullet \begin{bmatrix} 3s^2 & 2s & 1 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$\frac{\partial}{\partial t}Q(s,t) = \frac{\partial}{\partial t}\left(T^{\mathrm{T}}\right) \bullet M^{\mathrm{T}} \bullet \boldsymbol{G} \bullet M \bullet S$$

$$= \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix}^{\mathrm{T}} \bullet M^{\mathrm{T}} \bullet \boldsymbol{G} \bullet M \bullet S$$

$$\frac{\partial}{\partial s}Q(s,t) \times \frac{\partial}{\partial t}Q(s,t) \longleftarrow \quad \textbf{normal vector}$$