Homework #3
Due Date:
Reading Assignment: Chapter 3
Problems:

1. For what positive values of x are the following equations true?

(i) $\lfloor \ln x \rfloor = \lfloor \ln \lfloor x \rfloor \rfloor$

(ii) $\lfloor 2^x \rfloor = \lfloor 2^{\lfloor x \rfloor} \rfloor$

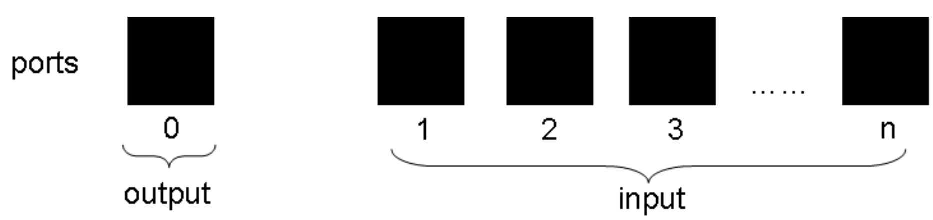
(iii) $\lfloor \log_2 x \rfloor = \lfloor \log_2 \lfloor x \rfloor \rfloor$

2. 3–17

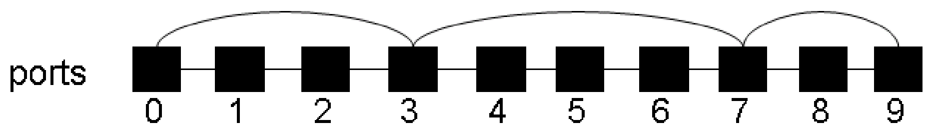
3. 3–28 (but use ceiling instead of floor)

4. This problem asks you to give an explicit expression for $A(n)$ in terms of n, where $A(n)$ is defined by a recurrence relation. Before we get to the recurrence, let us spend a little time explaining where it comes from. This is for motivation only; you are not responsible for it.

Suppose that we have an integrated circuit "chip" with $n+1$ "ports" arranged horizontally—one output port and n input ports, as pictured below:



The object is to wire the inputs to the output in order to transfer the information as quickly as possible, under the following restriction: In one time unit, information can flow between any two ports connected by a wire. All wires must run horizontally on one of the two levels.



One the bottom level, each pair of adjacent ports is connected by a wire. On the top level, the wires may "skip over" ports, but no two wires can cross (or else there would be a short circuit). Many items of information can flow simultaneously in both directions.

For example, in the figure below, there are 10 ports, labelled 0 through 9. Adjacent ports are connected together on the bottom level. There are three top level wires; they connect ports 0 & 3, 3 & 7, and 7 & 9. We can't add another top level wire between ports 2 and 6, because then the top level wires would cross.

This hookup allows information to be transferred from the inputs to the output in 3 time units. (That is, there is a path of length ≤ 3 from each input port to the output port.) There are other hookups that can do as well, but none are faster.

Let $A(n)$ be defined as the number of time units needed to transfer information from n input ports to the output port. We have just indicated that $A(9) = 3$. Now let's derive a recurrence for $A(n)$. Suppose we have a hookup that connects the output port to port k along the top level. Then there is a path from each of ports 1, 2, $\ldots$, k to the output port with length $\leq 1 + \lfloor \frac{k-1}{2} \rfloor$.

To get the shortest paths from ports $k+1$, $k+2$, $\ldots$, n to the output, we first find the shortest path to port k and then follow the top level wire to the output. We are allowed to use the top level, since it won't interfere with the wire from the output port to port k. Therefore, hooking up ports $k+1$, $k+2$, $\ldots$, n to port k is a subproblem of size $n-k$. (That is, regard port k as the output port and renumber the other ports by 1, 2, $\ldots$, $n-k$.) The fastest hookup takes $A(n-k)$ time units. The final path from port k to the output takes one unit, so the total time is $1 + A(n-k)$.

Choosing the best value of k in the above procedure, we get

$$A(n) = 1 + \min_{1 \le k \le n} \Big\{ \max\big(\lfloor \frac{k-1}{2} \rfloor, A(n-k)\big) \Big\}$$

Assume that $A(0) = 0$.
Find a simple expression for $A(n)$ in terms of n. (This is like Ex. 3–23.)

Note: For $n = 0$, 1, 2, 3, ..., the values of $A(n)$ are $0, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, \ldots$, namely, the sequence $A(n)$ includes two 1's, four 2's, six 3's, eight 4's, ten 5's, etc. There is no need to prove this property in this problem. Just derive an expression for $A(n)$.

5. (**Prob. 4** continued) Find a simple expression for any one of the values k that minimize

$$\max\big(\lfloor \frac{k-1}{2} \rfloor, A(n-k)\big)$$

(This would be important to a chip manufacturer who wants to know where to connect the top level wires.)

6. (**Prob. 5** continued) Show that

$$A(n) = 0, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, \ldots,$$

namely, there are two 1's, four 2's, six 3's, eight 4's, ten 5's, etc.