Mathematical Analysis of Algorithms

Homework #4
Due Date:
Reading Assignment: Chapter 4
Problems:

1. Prove that $\displaystyle\sum_{k|m} \frac{1}{k} \sum_{d|k} \mu(\frac{k}{d}) c^d = \frac{1}{m} \sum_{d|m} \varphi(d) c^{\frac{m}{d}}$

2. 4–29

3. 4–38

4. 4–41

5. The object of this problem is to determine the average execution time of the Insertion Sort algorithm. Suppose we want to sort n integers stored in an array $X[0\,..\,n]$. We assume that $X[0]$ contains garbage and that the n integers are stored as $X[1], X[2], \ldots, X[n]$.

Insertion sort is the method commonly used by people to sort bridge hands: consider the integers one at a time and insert each in its proper place among those already considered. The following program does exactly that.

#times executed	
n	**for** $i := 2$ **to** n **do begin**
$n-1$	$value := X[i]$;
$n-1$	$place := i-1$;
$A_n + n - 1$	**while** ($place \geq 1$ **and** $X[place] \geq value$) **do begin**
A_n	$X[place+1] := X[place]$;
A_n	$place := place - 1$;
	end;
$n-1$	$X[place+1] := value$;
	end;

Then number of times each statement is executed is given in the left column. Assume each statement takes one time unit to execute, except for the **while** statement. It takes two time units to execute, because there are two logical expressions to evaluate. The total running time of this algorithm is just

$$4A_n + 6n - 5.$$

To find the average running time, it suffices to find the average value of A_n, the number of times the inner loop is executed.

Assume that the n integers are $\{1, 2, \ldots, n\}$ and that all $n!$ permutations of the n integers are equally likely. The average (or expected) value of A_n is $E(A_n) = B_n/n!$, where

$$B_n = \sum_{\substack{\text{each permutation} \\ X[1], X[2], \ldots, X[n] \\ \text{of } \{1,2,\ldots,n\}}} \begin{pmatrix} \text{value of } A_n \text{ when the program} \\ \text{is run on input} X[1], \ldots, X[n] \end{pmatrix}.$$

For example, when $n = 3$, assume that we want to sort the integers $\{1, 2, 3\}$. For the 3! possible permutations, we get

$X[1]$	$X[2]$	$X[3]$	A_3
1	2	3	0
1	3	2	1
2	1	3	1
2	3	1	2
3	1	2	2
3	2	1	3

Therefore, $B_3 = 0 + 1 + 1 + 2 + 2 + 3 = 9$, so $E(A_3) = 9/3! = 3/2$.

(a) Find a simple expression for $E(A_n)$ in terms of n. Do that by deriving a recurrence relation for B_n in terms of B_{n-1}, then solving it by using the appropriate "summation factor." Compute the average running time $4(E(A_n)) + 6n - 5$.

(b) One way to speed up the algorithm is to get rid of the "*place* $\geq$ 1" expression, since it is seldom needed. To do that, we must initialize $X[0]$ to $-\infty$. Now the "**while**" statement takes only one time unit to execute. Does this alteration affect the value of A_n? Why or why not? What is the resulting average running time of the algorithm?

(c) Is this algorithm *stable*? If not, can you modify the algorithm to make it stable? A sorting method is *stable* if equal keys remain in the same relative order in the sorted sequence as they were in originally.