Homework #9
Due Date:
Reading Assignment: 9.1–9.4
Problems:

1. 8–31

(a) Solve the problem by first giving the recurrence relations of A_n, B_n, C_n, D_n, and E_n, and then finding $A(z)$, the generating function of A_n. The term A_n stands for the probability that the worm is at vertex A after n days, and so forth.

(b) Redo the problem by first finding the equations of regular expressions A, B, C, D, and E, where A is the set of all paths lead the worm to vertex A, and so forth. You may merge some of the sets to simplify the derivation.

2. 9–2

3. 9–13

4. 9–14

5. 9–34

6. Let Y_n be the random variable describing the number of the probes for unsuccessful search on a binary search tree that contains n keys. The purpose of this problem is to compute $E(Y_n)$ and $Var(Y_n)$ using generating functions. The generating function for Y_n is defined by

$$G_n(Z) = \sum_k p_{nk} z^k,$$

where $p_{nk} = \Pr(Y_n = k)$, the probability that k probes are needed to do an unsuccessful search in an n-key tree. The sample space for Y_n is

$$S = \{(x_1, \ldots, x_{n-1}, x_n; y)\},$$

where each $(x_1, \ldots, x_{n-1}, x_n; y)$ is one of the $(n+1)!$ permutations of 1, 2, …, $n+1$. Here $x_1, \ldots, x_n$ represents the n keys already inserted, and y is the key that will be searched for unsuccessfully.

We want to develop a recurrence for p_{nk} by deriving a relationship between Y_n and Y_{n-1}. To do this, we use the model described in class: Another way to think of $s = (x_1, \ldots, x_{n-1}, x_n; y)$ is to regard $s' = (x_1, \ldots, x_{n-1}; y)$ as one of the $n!$ permutations of $1, 2, \ldots, n$, and to regard x_n as one of the $n+1$ fractions $\frac{1}{2}, \frac{3}{2}, \ldots, (n+\frac{1}{2})$.

(a) Show that

$$Y_n(s) = \begin{cases} Y_{n-1}(s') + 1 & \text{when } x_n = y \pm \frac{1}{2}, \\ Y_{n-1}(s') & \text{when } x_n \text{ is one of the other } n-1 \text{ values.} \end{cases}$$

(This isn't hard if you think about it.) Using this key fact, we have

$$\begin{aligned} p_{nk} = {} & \Pr\Big(Y_{n-1} = k-1 \text{ and } \big(x_n = y - \frac{1}{2} \text{ or } x_n = y + \frac{1}{2}\big)\Big) \\ & + \Pr\Big(Y_{n-1} = k \text{ and } x_n \text{ is one of the other } n-1 \text{ values}\Big). \end{aligned}$$

(b) Using independence, derive a recurrence for p_{nk} in terms of $p_{n-1,k}$ and $p_{n-1,k-1}$. Add the appropriate δ term to make the recurrence hold for all values of n and k and substitute the recurrence into the definition of $G_n(z)$. Compute $E(Y_n)$ and $Var(Y_n)$. As a check, the variance is

$$Var(Y_n) = 2H_{n+1} - 4H_{n+1}^{(2)} + 2.$$

(c) Express $E(Y_n)$ and $Var(Y_n)$ in the form $f(n) + c + g(n) + O(1/n)$, where c is a constant and $f(n)$ and $g(n)$ are some elementary functions. (e.g., n, e^n, $\log n$, $\log n/n$).