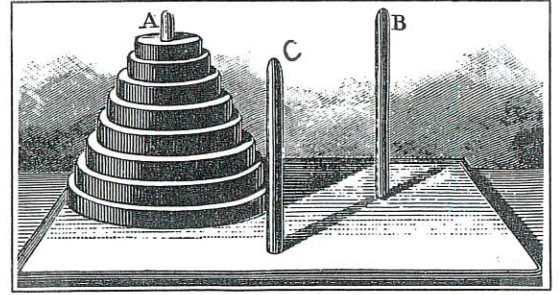


Chap 1. Recurrence

No. _____
Date: / /

• How to solve a problem

- (1) Small cases (get a feel for the prob)
- (2) Find a pattern
- (3) Guess the solution
- (4) Prove/derive the solution



• Example 1 (Tower of Hanoi)

n	1	2	3	4	5	6	7
T _n	1	3	7	15	31	63	127

$$\begin{cases} T_1 = 1 \\ T_n = 2T_{n-1} + 1 \end{cases} \Rightarrow \text{(甲)} T_n = 2^n - 1 \quad (\text{by induction})$$

$$\text{(乙)} T_{n+1} = 2(T_n + 1) = 2^2(T_{n-2} + 1) = \dots = 2^{n-1}(T_1 + 1) = 2^n$$

$$\text{(丙)} \frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$$

$$\frac{T_{n-1}}{2^{n-1}} = \frac{T_{n-2}}{2^{n-2}} + \frac{1}{2^{n-1}} \Rightarrow T_n = 2^n \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \dots + \frac{1}{2} \right)$$

$$\frac{T_2}{2^2} = \frac{T_1}{2^1} + \frac{1}{2^2} \Rightarrow T_n = 1 + 2 + \dots + 2^{n-1} = 2^n - 1$$

• Example 2 (Josephus problem)

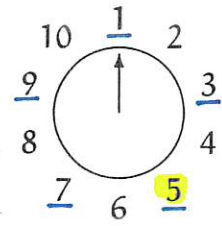
(1) J(n) 奇數 (Masada) n=41

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

$$(2) \begin{cases} J(2^m) = 1 \\ J(2^m - 1) = 2^m - 1 \end{cases}$$

$$(3) \begin{cases} J(2n) = 2J(n) - 1, & J(10) = 2 \cdot 3 - 1 \\ J(2n+1) = 2J(n) + 1, & J(11) = 2 \cdot 3 + 1 \end{cases}$$

n = 10

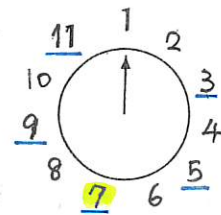


$$\begin{aligned} 1 &= 2 \cdot 1 - 1 \\ 3 &= 2 \cdot 2 - 1 \\ 5 &= 2 \cdot 3 - 1 \\ 7 &= 2 \cdot 4 - 1 \\ 9 &= 2 \cdot 5 - 1 \end{aligned}$$

$$\Rightarrow J(n) = J(2^m + l) = 2l + 1 = 2(n - 2^{\lfloor \lg n \rfloor}) + 1$$

$$= J(\underbrace{b_m b_{m-1} \dots b_0}_1)_2 = (b_{m-1} \dots b_0)_2$$

n = 11



$$\begin{aligned} 3 &= 2 \cdot 1 + 1 \\ 5 &= 2 \cdot 2 + 1 \\ 7 &= 2 \cdot 3 + 1 \\ 9 &= 2 \cdot 4 + 1 \\ 11 &= 2 \cdot 5 + 1 \end{aligned}$$

$$\begin{cases} J(10) = J(1010)_2 = (101)_2 = 5 \\ J(100) = J(1100100)_2 = (1001001)_2 = 73 \end{cases}$$

問題 J(n) = n/2, n=?

(1) 奇數

$$n = 2 = \quad \quad \quad 10$$

$$10 = \quad \quad \quad 1010$$

$$42 = \quad \quad \quad 101010$$

$$170 = \quad \quad \quad 10101010$$

$$(1) 2l+1 = \frac{2^m+l}{2} \Rightarrow m \text{ 奇數 (否則 } m=2k, (3+1)^k = 3l+2 \text{ (x))}$$

$$(2) \frac{n}{2} = 1 \quad b_6 \quad b_5 \quad b_4 \quad b_3 \quad b_2 \quad b_1 \quad 0$$

$$\frac{n}{2} = \quad \quad \quad 1 \quad b_6 \quad b_5 \quad b_4 \quad b_3 \quad b_2 \quad b_1$$

$$J(n) = \quad \quad \quad b_6 \quad b_5 \quad b_4 \quad b_3 \quad b_2 \quad b_1 \quad 0 \quad 1$$

Generalization

(I) (recurrence, period=2) (Josephus: $\alpha=1$, $\beta=-1$, $\gamma=1$)

(II) (period=3)

$$\begin{cases} f(1) = \alpha \\ f(2n) = 2f(n) + \beta \quad (n \geq 1) \\ f(2n+1) = 2f(n) + \gamma \end{cases}$$

$$\begin{cases} f(1) = 34 \\ f(2) = 5 \\ f(3n) = 10f(n) + 76 \quad (n \geq 1) \\ f(3n+1) = 10f(n) - 2 \\ f(3n+2) = 10f(n) + 8 \end{cases}$$

n	f(n)	f(n)	n
1	α	α	1
2	$2\alpha + \beta$	$2\alpha + \beta$	10
3	$2\alpha + \gamma$	$2\alpha + \gamma$	11
4	$4\alpha + 3\beta$	$4\alpha + 2\beta + \beta$	100
5	$4\alpha + 2\beta + \gamma$	$4\alpha + 2\beta + \gamma$	101
6	$4\alpha + \beta + 2\gamma$	$4\alpha + 2\gamma + \beta$	110
7	$4\alpha + 3\gamma$	$4\alpha + 2\gamma + \gamma$	111
8	$8\alpha + 7\beta$		
9	$8\alpha + 6\beta + \gamma$		
10	$8\alpha + 5\beta + 2\gamma$	$8\alpha + 4\beta + 2\gamma + \beta$	1010
11	$8\alpha + 4\beta + 3\gamma$		

$$\begin{aligned} f(46) &= f(1201)_3 \\ &= 10f(120)_3 - 2 \\ &= 10^2 f(12)_3 + 10 \cdot 76 - 2 \\ &= 10^3 f(1)_3 + 10^2 \cdot 8 + 10 \cdot 76 - 2 \\ &= 10^3 \cdot 34 + 10^2 \cdot 8 + 10 \cdot 76 - 2 \\ &= (34, 8, 76, -2)_{10} \end{aligned}$$

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$$\begin{cases} A(n) = 2^m & \text{(Josephus)} \\ B(n) = 2^m - 1 - l & \xrightarrow[\gamma=1]{\alpha=1, \beta=-1} J(n) = 2l + 1 \\ C(n) = l \end{cases}$$

(1) Induction:

$$(1+2+\dots+n = An^2 + Bn + C) \quad n=0, 1, 2$$

(2) Repertoire:

(α, β, γ)	$f(n)$	Eg.
$(1, 0, 0)$	2^m	$A(n) = 2^m$
$(1, -1, -1)$	1	$A(n) - B(n) - C(n) = 1$
$(1, 0, 1)$	n	$A(n) + C(n) = n$

$$\begin{cases} f(1) = 1 \\ f(2n) = 2f(n) \\ f(2n+1) = 2f(n) \end{cases} \quad f(n) = 2^m$$

$$\begin{cases} f(n) = 1 \\ 1 = \alpha \\ 1 = 2 + \beta \\ 1 = 2 + \gamma \end{cases} \quad (\alpha, \beta, \gamma) = (1, -1, -1)$$

$$\begin{aligned} (3) f(1010)_2 &= 2f(101) + \beta \\ &= 2^2 f(10) + 2\gamma + \beta \\ &= 2^3 f(1) + 2^2 \beta + 2\gamma + \beta \\ &= 2^3 \alpha + 2^2 \beta + 2\gamma + \beta \\ &= (\alpha \beta \gamma \beta)_2 \end{aligned}$$

$$\begin{cases} f(n) = n \\ 1 = \alpha \\ 2n = 2n + \beta \\ 2n+1 = 2n + \gamma \end{cases} \quad (\alpha, \beta, \gamma) = (1, 0, 1)$$