Chap3 Integer functions

$\begin{cases} \lfloor 3.6 \rfloor = 3, & \lfloor -3.6 \rfloor = -4, \\ \lceil 3.6 \rceil = 4, & \lceil -3.6 \rceil = -3, \end{cases}$ $\begin{cases} n = \left\lfloor \frac{n}{m} \right\rfloor m + n \bmod m \\ \lfloor n+x \rfloor = n + \lfloor x \rfloor \quad n \in \mathbb{Z} \end{cases}$

• **Example 1** (p.75) $\begin{cases} +5 & \lfloor \sqrt[3]{n} \rfloor \mid n \quad (1 \le n \le 10^3) \\ -1 & \text{else} \end{cases}$

n	$1 \cdots 7$	$8 \cdots 26$	$27 \cdots 63$	$k^3 \cdots (k^3+3k^2+3k)$	$(k+1)^3 \cdots$	10^3
$\lfloor \sqrt[3]{n} \rfloor$	$1 \cdots 1$	$2 \cdots 2$	$3 \cdots 3$	$k \cdots k$	$k+1$	10
	7	10	13	$3k+4$		

$$W = \sum_{1 \le k \le 9} (3k+4) + 1 = 172$$

(期望值) $E = (+5)\frac{172}{1000} + (-1)\frac{828}{1000} = \frac{32}{1000}$

• **Example 2** (p.86) 求 $\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$

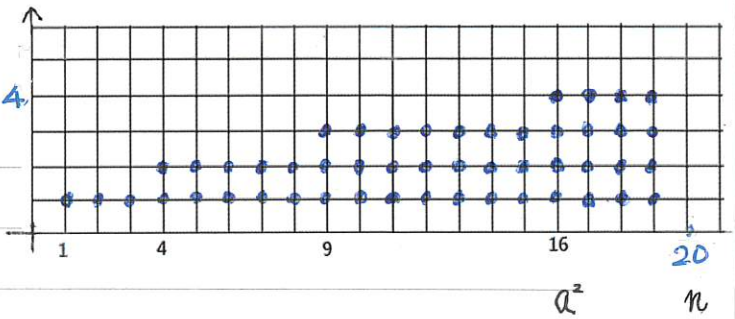
k	0	$1\ 2\ 3$	$4\ 5 \cdots 8$	$9 \cdots 15$	$16 \cdots$	$m^2 \cdots m^2+2m$	$(m+1)^2 \cdots$	$a^2 \cdots n$
$\lfloor \sqrt{k} \rfloor$	0	$1\ 1\ 1$	$2\ 2 \cdots 2$	$3 \cdots 3$	$4 \cdots$	$m \cdots m$	$m+1 \cdots$	$a \cdots a$
		3	5	7	$\cdots$	$2m+1$		$a = \lfloor \sqrt{n} \rfloor$

$$= \sum_{0 \le m < a} m(2m+1) + a(n-a^2)$$

$$= \frac{2(a-1)a(2a-1)}{6} + \frac{a(a-1)}{2} + na - a^3$$

$$= na - \frac{1}{6}a(a+1)(2a+1)$$

$$= na - (1^2+2^2+\cdots+a^2)$$



• **Example 3** (p.90) 求 $f(m,n,x) = \sum_{0 \le k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{n+x}{m} \right\rfloor + \left\lfloor \frac{2n+x}{m} \right\rfloor + \cdots + \left\lfloor \frac{(m-1)n+x}{m} \right\rfloor$

(1) $(n=1)$ $f(m,1,x) = \left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x+1}{m} \right\rfloor + \left\lfloor \frac{x+2}{m} \right\rfloor + \cdots + \left\lfloor \frac{x+(m-1)}{x} \right\rfloor = \lfloor x \rfloor$ (Hermite 等式)

$(m=5)$ $f(5,1,\pi) = \left\lfloor \frac{\pi}{5} \right\rfloor + \left\lfloor \frac{\pi+1}{5} \right\rfloor + \left\lfloor \frac{\pi+2}{5} \right\rfloor + \left\lfloor \frac{\pi+3}{5} \right\rfloor + \left\lfloor \frac{\pi+4}{5} \right\rfloor = 3$

$f(5,1,-8.3) = \left\lfloor \frac{-8.3}{5} \right\rfloor + \left\lfloor \frac{-7.3}{5} \right\rfloor + \left\lfloor \frac{-6.3}{5} \right\rfloor + \left\lfloor \frac{-5.3}{5} \right\rfloor + \left\lfloor \frac{-4.3}{5} \right\rfloor$

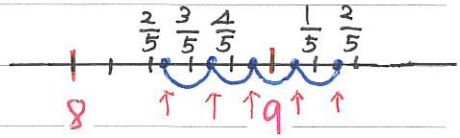
$= (-2) + (-2) + (-2) + (-2) + (-1) = -9$

$f(5,1,42.3) = \left\lfloor \frac{42.3}{5} \right\rfloor + \left\lfloor \frac{43.3}{5} \right\rfloor + \left\lfloor \frac{44.3}{5} \right\rfloor + \left\lfloor \frac{45.3}{5} \right\rfloor + \left\lfloor \frac{46.3}{5} \right\rfloor$

$= \left\lfloor 8+\frac{2}{5}+\frac{0.3}{5} \right\rfloor + \left\lfloor 8+\frac{3}{5}+\frac{0.3}{5} \right\rfloor + \left\lfloor 8+\frac{4}{5}+\frac{0.3}{5} \right\rfloor + \left\lfloor 8+\frac{5}{5}+\frac{0.3}{5} \right\rfloor + \left\lfloor 8+\frac{6}{5}+\frac{0.3}{5} \right\rfloor$

$= 8 + 8 + 8 + (8+1) + (8+1)$

$= 5 \cdot 8 + 2 = 42$



• $x = mp + q + r$, $\begin{cases} 0 \le q < m \\ 0 \le r < 1 \end{cases}$

$$\left\lfloor \frac{x+k}{m} \right\rfloor = \left\lfloor p + \frac{q+k}{m} + \frac{r}{m} \right\rfloor = p + \left\lfloor \frac{q+k}{m} \right\rfloor$$

$$\Rightarrow f(m,1,x) = \sum_{0 \le k \le m-1} \left\lfloor \frac{x+k}{m} \right\rfloor = mp + q = \lfloor x \rfloor$$

$\begin{cases} m = m_0 d, \\ n = n_0 d, \end{cases} (m,n) = d, \quad m_0 \perp n_0$

$\mathbb{Z}_m = \{0, 1, 2, \dots, m-1\}$, $\mathbb{Z}_m * n \equiv \{0, 1d, 2d, \cdots, (m_0-1)d\}^d$

$\begin{cases} m = 12 = 4 \cdot 3, \\ n = 9 = 3 \cdot 3, \end{cases}$ $(12,9) = 3$

$\mathbb{Z}_{12} = \{0, 1, 2, 3, \dots, 11\}$,

$\mathbb{Z}_{12} * 5 = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55\} \bmod 12$

$\equiv \{0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7\}$

$\mathbb{Z}_{12} * 9 = \{0, 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99\} \bmod 12$

$\equiv \{0,9,6,3,\ 0,9,6,3,\ 0,9,6,3\}$

$= \{0,9,6,3\}^3$

(2) $f(m,n,x) = \sum_{0 \le k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{n+x}{m} \right\rfloor + \left\lfloor \frac{2n+x}{m} \right\rfloor + \cdots + \left\lfloor \frac{(m-1)n+x}{m} \right\rfloor$

$f(12,9,x) = \left\lfloor \frac{x}{12} \right\rfloor + \left\lfloor \frac{9+x}{12} \right\rfloor + \left\lfloor \frac{18+x}{12} \right\rfloor + \left\lfloor \frac{27+x}{12} \right\rfloor + \left\lfloor \frac{36+x}{12} \right\rfloor + \left\lfloor \frac{45+x}{12} \right\rfloor + \left\lfloor \frac{54+x}{12} \right\rfloor + \left\lfloor \frac{63+x}{12} \right\rfloor + \left\lfloor \frac{72+x}{12} \right\rfloor + \left\lfloor \frac{81+x}{12} \right\rfloor + \left\lfloor \frac{90+x}{12} \right\rfloor + \left\lfloor \frac{99+x}{12} \right\rfloor$

$= \left\lfloor \frac{0}{12} \right\rfloor + \left\lfloor \frac{9}{12} \right\rfloor + \left\lfloor \frac{18}{12} \right\rfloor + \left\lfloor \frac{27}{12} \right\rfloor + \left\lfloor \frac{36}{12} \right\rfloor + \left\lfloor \frac{45}{12} \right\rfloor + \left\lfloor \frac{54}{12} \right\rfloor + \left\lfloor \frac{63}{12} \right\rfloor + \left\lfloor \frac{72}{12} \right\rfloor + \left\lfloor \frac{81}{12} \right\rfloor + \left\lfloor \frac{90}{12} \right\rfloor + \left\lfloor \frac{99}{12} \right\rfloor$

$+ \left\lfloor \frac{x}{12} \right\rfloor + \left\lfloor \frac{9+x}{12} \right\rfloor + \left\lfloor \frac{6+x}{12} \right\rfloor + \left\lfloor \frac{3+x}{12} \right\rfloor + \left\lfloor \frac{x}{12} \right\rfloor + \left\lfloor \frac{9+x}{12} \right\rfloor + \left\lfloor \frac{6+x}{12} \right\rfloor + \left\lfloor \frac{3+x}{12} \right\rfloor + \left\lfloor \frac{x}{12} \right\rfloor + \left\lfloor \frac{9+x}{12} \right\rfloor + \left\lfloor \frac{6+x}{12} \right\rfloor + \left\lfloor \frac{3+x}{12} \right\rfloor$

$= \left(\frac{0-0}{12} + \frac{9-9}{12} + \frac{18-6}{12} + \frac{27-3}{12} + \frac{36-0}{12} + \frac{45-9}{12} + \frac{54-6}{12} + \frac{63-3}{12} + \frac{72-0}{12} + \frac{81-9}{12} + \frac{90-6}{12} + \frac{99-3}{12} \right) + 3\left(\left\lfloor \frac{x}{12} \right\rfloor + \left\lfloor \frac{3+x}{12} \right\rfloor + \left\lfloor \frac{6+x}{12} \right\rfloor + \left\lfloor \frac{9+x}{12} \right\rfloor \right)$

$= \frac{0+9+18+\cdots+99}{12} - 3\,\frac{0+3+6+9}{12} + 3\left(\left\lfloor \frac{x/3}{4} \right\rfloor + \left\lfloor \frac{1+x/3}{4} \right\rfloor + \left\lfloor \frac{2+x/3}{4} \right\rfloor + \left\lfloor \frac{3+x/3}{4} \right\rfloor \right)$ $\quad 3\lfloor x/3 \rfloor$

$= \sum_{0 \le k < 12} \frac{9k}{12} - 3 \sum_{0 \le k < 4} \frac{3k}{12} + 3 \sum_{0 \le k < 4} \left\lfloor \frac{k+x/3}{4} \right\rfloor$

$= \frac{n}{m} \frac{m(m-1)}{2} - d\,\frac{d}{m} \frac{m_0(m_0-1)}{2} + d \left\lfloor \frac{x}{d} \right\rfloor$

$= \frac{(m-1)(n-1)}{2} + \frac{d-1}{2} + d \left\lfloor \frac{x}{d} \right\rfloor$

$f(m,n,x) = \sum_{0 \le k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor$

$= \sum_{0 \le k < m} \left\lfloor \frac{\lfloor \frac{nk}{m} \rfloor m + nk \bmod m + x}{m} \right\rfloor$

$= \sum_{0 \le k < m} \left\lfloor \frac{nk}{m} \right\rfloor + \sum_{0 \le k < m} \left\lfloor \frac{nk \bmod m + x}{m} \right\rfloor$

(*) $= \sum_{0 \le k < m} \left(\frac{nk - nk \bmod m}{m} \right) + \sum_{0 \le k < m} \left\lfloor \frac{nk \bmod m + x}{m} \right\rfloor$

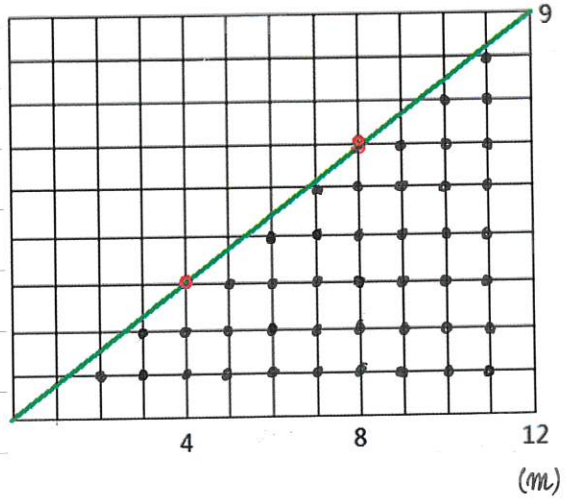
$= \frac{n}{m} \sum_{0 \le k < m} k - \frac{1}{m} d \sum_{0 \le k < m_0} kd + d \sum_{0 \le k < m_0} \left\lfloor \frac{kd+x}{m} \right\rfloor$

$= \frac{n}{m} \frac{m(m-1)}{2} - \frac{d \cdot d\, m_0(m_0-1)}{m} \frac{1}{2} + d \sum_{0 \le k < m_0} \left\lfloor \frac{k + x/d}{m_0} \right\rfloor$

$= \frac{n(m-1)}{2} - \frac{m-d}{2} + d \left\lfloor \frac{x}{d} \right\rfloor$

$= \frac{(m-1)(n-1)}{2} + \frac{d-1}{2} + d \left\lfloor \frac{x}{d} \right\rfloor$

$y = \frac{9}{12} x$



$y = \frac{n}{m} x, \quad (m,n) = d$

(*) $= \frac{(m-1)(n-1) - (d-1)}{2} + (d-1)$

$= \frac{(m-1)(n-1)}{2} + \frac{d-1}{2}$

(Quadratic reciprocity)