Stirling numbers

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- **1st kind:** $\begin{bmatrix} n \\ k \end{bmatrix}$:= # of n–permutations with k cycles. (n cycle k)
  (beads) (necklaces)

  2nd kind: $\begin{Bmatrix} n \\ k \end{Bmatrix}$:= # of partitions of an n–element set into k subsets. (n subset k)

- $\begin{Bmatrix} 4 \\ 2 \end{Bmatrix} = 7$, $\{1,2,3,4\} =$ $\{1,2,3\}\cup\{4\}$, $\{1,2,4\}\cup\{3\}$, $\{1,3,4\}\cup\{2\}$, $\{2,3,4\}\cup\{1\}$, $\{1,2\}\cup\{3,4\}$, $\{1,3\}\cup\{2,4\}$, $\{1,4\}\cup\{2,3\}$;

  $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$, $[1,2,3]\,[4]$, $[1,2,4]\,[3]$, $[1,3,4]\,[2]$, $[2,3,4]\,[1]$, $[1,3,2]\,[4]$, $[1,4,2]\,[3]$, $[1,4,3]\,[2]$, $[2,4,3]\,[1]$, $[1,2]\,[3,4]$, $[1,3]\,[2,4]$, $[1,4]\,[2,3]$;

- Recurrences

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}.$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Permutation (bijective functions)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 8 & 4 & 7 & 2 & 9 & 1 & 5 & 6 & 10 \end{pmatrix} \equiv [1,3,4,7]\,[2,8,5]\,[6,9]\,[10]$$

$\uparrow$ 11

- Special values ($n>0$)

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} = 0$$

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix} = 1 \qquad \begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$$

$$\begin{Bmatrix} n \\ 2 \end{Bmatrix} = (2^{n-1}-1) \qquad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\,H_{n-1}$$

$$\begin{Bmatrix} n \\ n-1 \end{Bmatrix} = \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}.$$

$$\begin{Bmatrix} n \\ n \end{Bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = \binom{n}{n} = 1.$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k} = 0, \quad \text{if } k > n.$$

n	$\begin{Bmatrix} n \\ 0 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 2 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 3 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 4 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 5 \end{Bmatrix}$	$\begin{Bmatrix} n \\ 6 \end{Bmatrix}$
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

n	$\begin{bmatrix} n \\ 0 \end{bmatrix}$	$\begin{bmatrix} n \\ 1 \end{bmatrix}$	$\begin{bmatrix} n \\ 2 \end{bmatrix}$	$\begin{bmatrix} n \\ 3 \end{bmatrix}$	$\begin{bmatrix} n \\ 4 \end{bmatrix}$	$\begin{bmatrix} n \\ 5 \end{bmatrix}$	$\begin{bmatrix} n \\ 6 \end{bmatrix}$
0	1						
1	0	1					
2	0	1	1				
3	0	2	3	1			
4	0	6	11	6	1		
5	0	24	50	35	10	1	
6	0	120	274	225	85	15	1

- $\begin{Bmatrix} n \\ 2 \end{Bmatrix} = \frac{1}{2}\left[\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}\right] = \frac{1}{2}\left[2^n-2\right]$ #

  $[n, a_{n-1}, \ldots, a_{k+1}][a_k, a_{k-1}, \ldots, a_1]$

- $\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ 2 \end{bmatrix} + (n-2)!$

  或 $\begin{bmatrix} n \\ 2 \end{bmatrix} = \sum_{1\le k\le n-1} (n-1)\cdots(k+1)\,\frac{k(k-1)\cdots 1}{k}$

  $= (n-1)!\sum_{1\le k\le n-1}\frac{1}{k} = (n-1)!\,H_{n-1}$

$$\frac{\begin{bmatrix} n \\ 2 \end{bmatrix}}{(n-1)!} = \frac{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}{(n-2)!} + \frac{1}{n-1} = \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}{1!}$$

$$= H_{n-1}$$

$\begin{cases} x \text{ 色} \\ 4 \text{ 房} \end{cases}$

- Converting between powers:

$$x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}} = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}}.$$

$$x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k;$$

$$x^{\overline{n}} = \sum_k \left[{n \atop k} \right] x^k.$$

$$x^4 = x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}$$
$$= \left\{ {4 \atop 4} \right\} x^{\underline{4}} + \left\{ {4 \atop 3} \right\} x^{\underline{3}} + \left\{ {4 \atop 2} \right\} x^{\underline{2}} + \left\{ {4 \atop 1} \right\} x^{\underline{1}}$$

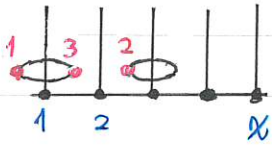
$$x^{\underline{4}} = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$$
$$= \left[{4 \atop 4} \right] x^4 - \left[{4 \atop 3} \right] x^3 + \left[{4 \atop 2} \right] x^2 - \left[{4 \atop 1} \right] x$$

$$x^{\overline{4}} = x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x$$
$$= \left[{4 \atop 4} \right] x^4 + \left[{4 \atop 3} \right] x^3 + \left[{4 \atop 2} \right] x^2 + \left[{4 \atop 1} \right] x$$

- Inversion formulas:

$$\sum_k \left[{n \atop k} \right] \left\{ {k \atop m} \right\} (-1)^{n-k} = [m=n];$$

$$\sum_k \left\{ {n \atop k} \right\} \left[{k \atop m} \right] (-1)^{n-k} = [m=n].$$



$\begin{cases} 4 \text{ 珍珠} \\ x \text{ 釘柱} \end{cases}$

$$\begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x^1 \\ x^0 \end{bmatrix} = \begin{bmatrix} \left\{ {4 \atop 4} \right\} & \left\{ {4 \atop 3} \right\} & \left\{ {4 \atop 2} \right\} & \left\{ {4 \atop 1} \right\} & \left\{ {4 \atop 0} \right\} \\ 0 & \left\{ {3 \atop 3} \right\} & \left\{ {3 \atop 2} \right\} & \left\{ {3 \atop 1} \right\} & \left\{ {3 \atop 0} \right\} \\ 0 & 0 & \left\{ {2 \atop 2} \right\} & \left\{ {2 \atop 1} \right\} & \left\{ {2 \atop 0} \right\} \\ 0 & 0 & 0 & \left\{ {1 \atop 1} \right\} & \left\{ {1 \atop 0} \right\} \\ 0 & 0 & 0 & 0 & \left\{ {0 \atop 0} \right\} \end{bmatrix} \begin{bmatrix} x^{\underline{4}} \\ x^{\underline{3}} \\ x^{\underline{2}} \\ x^{\underline{1}} \\ x^{\underline{0}} \end{bmatrix}, \quad \begin{bmatrix} x^{\overline{4}} \\ x^{\overline{3}} \\ x^{\overline{2}} \\ x^{\overline{1}} \\ x^{\overline{0}} \end{bmatrix} = \begin{bmatrix} \left[{4 \atop 4} \right] & \left[{4 \atop 3} \right] & \left[{4 \atop 2} \right] & \left[{4 \atop 1} \right] & \left[{4 \atop 0} \right] \\ 0 & \left[{3 \atop 3} \right] & \left[{3 \atop 2} \right] & \left[{3 \atop 1} \right] & \left[{3 \atop 0} \right] \\ 0 & 0 & \left[{2 \atop 2} \right] & \left[{2 \atop 1} \right] & \left[{2 \atop 0} \right] \\ 0 & 0 & 0 & \left[{1 \atop 1} \right] & \left[{1 \atop 0} \right] \\ 0 & 0 & 0 & 0 & \left[{0 \atop 0} \right] \end{bmatrix} \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x^1 \\ x^0 \end{bmatrix}$$

$$\begin{bmatrix} x^{\underline{4}} \\ x^{\underline{3}} \\ x^{\underline{2}} \\ x^{\underline{1}} \\ x^{\underline{0}} \end{bmatrix} = \begin{bmatrix} \left[{4 \atop 4} \right] & -\left[{4 \atop 3} \right] & \left[{4 \atop 2} \right] & -\left[{4 \atop 1} \right] & \left[{4 \atop 0} \right] \\ 0 & \left[{3 \atop 3} \right] & -\left[{3 \atop 2} \right] & \left[{3 \atop 1} \right] & -\left[{3 \atop 0} \right] \\ 0 & 0 & \left[{2 \atop 2} \right] & -\left[{2 \atop 1} \right] & \left[{2 \atop 0} \right] \\ 0 & 0 & 0 & \left[{1 \atop 1} \right] & -\left[{1 \atop 0} \right] \\ 0 & 0 & 0 & 0 & \left[{0 \atop 0} \right] \end{bmatrix} \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x^1 \\ x^0 \end{bmatrix}$$

$$\begin{bmatrix} \left\{ {4 \atop 4} \right\} & \left\{ {4 \atop 3} \right\} & \left\{ {4 \atop 2} \right\} & \left\{ {4 \atop 1} \right\} & \left\{ {4 \atop 0} \right\} \\ 0 & \left\{ {3 \atop 3} \right\} & \left\{ {3 \atop 2} \right\} & \left\{ {3 \atop 1} \right\} & \left\{ {3 \atop 0} \right\} \\ 0 & 0 & \left\{ {2 \atop 2} \right\} & \left\{ {2 \atop 1} \right\} & \left\{ {2 \atop 0} \right\} \\ 0 & 0 & 0 & \left\{ {1 \atop 1} \right\} & \left\{ {1 \atop 0} \right\} \\ 0 & 0 & 0 & 0 & \left\{ {0 \atop 0} \right\} \end{bmatrix} \begin{bmatrix} \left[{4 \atop 4} \right] & -\left[{4 \atop 3} \right] & \left[{4 \atop 2} \right] & -\left[{4 \atop 1} \right] & \left[{4 \atop 0} \right] \\ 0 & \left[{3 \atop 3} \right] & -\left[{3 \atop 2} \right] & \left[{3 \atop 1} \right] & -\left[{3 \atop 0} \right] \\ 0 & 0 & \left[{2 \atop 2} \right] & -\left[{2 \atop 1} \right] & \left[{2 \atop 0} \right] \\ 0 & 0 & 0 & \left[{1 \atop 1} \right] & -\left[{1 \atop 0} \right] \\ 0 & 0 & 0 & 0 & \left[{0 \atop 0} \right] \end{bmatrix} = I_5$$

$$\begin{bmatrix} \left[{4 \atop 4} \right] & -\left[{4 \atop 3} \right] & \left[{4 \atop 2} \right] & -\left[{4 \atop 1} \right] & \left[{4 \atop 0} \right] \\ 0 & \left[{3 \atop 3} \right] & -\left[{3 \atop 2} \right] & \left[{3 \atop 1} \right] & -\left[{3 \atop 0} \right] \\ 0 & 0 & \left[{2 \atop 2} \right] & -\left[{2 \atop 1} \right] & \left[{2 \atop 0} \right] \\ 0 & 0 & 0 & \left[{1 \atop 1} \right] & -\left[{1 \atop 0} \right] \\ 0 & 0 & 0 & 0 & \left[{0 \atop 0} \right] \end{bmatrix} \begin{bmatrix} \left\{ {4 \atop 4} \right\} & \left\{ {4 \atop 3} \right\} & \left\{ {4 \atop 2} \right\} & \left\{ {4 \atop 1} \right\} & \left\{ {4 \atop 0} \right\} \\ 0 & \left\{ {3 \atop 3} \right\} & \left\{ {3 \atop 2} \right\} & \left\{ {3 \atop 1} \right\} & \left\{ {3 \atop 0} \right\} \\ 0 & 0 & \left\{ {2 \atop 2} \right\} & \left\{ {2 \atop 1} \right\} & \left\{ {2 \atop 0} \right\} \\ 0 & 0 & 0 & \left\{ {1 \atop 1} \right\} & \left\{ {1 \atop 0} \right\} \\ 0 & 0 & 0 & 0 & \left\{ {0 \atop 0} \right\} \end{bmatrix} = I_5$$

(6.15) $\underbrace{\{\cdots\}_1 \{\cdots\}_2 \cdots \{\cdots\}_m}_{k} \underbrace{\{\cdots,}_{n-k} 0\}_{m+1}$ $\quad \begin{cases} n=12 \\ m=2 \end{cases}$ $k=5$

(6.16) $[0\ 9\ 12\ 7\ 10\ 1\ 2\ 11]\underline{[3\ 5]}\underline{[4\ 8\ 6]} \Longleftrightarrow [1\ 2\ 11]\underline{[3\ 5]}\underline{[4\ 8\ 6]}[7\ 10][9\ 12]$

Table 265 Additional Stirling number identities, for integers $l, m, n \geqslant 0$.

$$\left\{ {n+1 \atop m+1} \right\} = \sum_k \binom{n}{k} \left\{ {k \atop m} \right\}. \tag{6.15}$$

$$\left[{n+1 \atop m+1} \right] = \sum_k \left[{n \atop k} \right] \binom{k}{m}. \tag{6.16}$$

$$\left\{ {n \atop m} \right\} = \sum_k \binom{n}{k} \left\{ {k+1 \atop m+1} \right\} (-1)^{n-k}. \tag{6.17}$$

$$\left[{n \atop m} \right] = \sum_k \left[{n+1 \atop k+1} \right] \binom{k}{m} (-1)^{m-k}. \tag{6.18}$$

$$m! \left\{ {n \atop m} \right\} = \sum_k \binom{m}{k} k^n (-1)^{m-k}. \tag{6.19}$$

$$\left\{ {n+1 \atop m+1} \right\} = \sum_{k=0}^{n} \left\{ {k \atop m} \right\} (m+1)^{n-k}. \tag{6.20}$$

$$\left[{n+1 \atop m+1} \right] = \sum_{k=0}^{n} \left[{k \atop m} \right] n^{\underline{n-k}} = n! \sum_{k=0}^{n} \left[{k \atop m} \right] / k!. \tag{6.21}$$

$$\left\{ {m+n+1 \atop m} \right\} = \sum_{k=0}^{m} k \left\{ {n+k \atop k} \right\}. \tag{6.22}$$

$$\left[{m+n+1 \atop m} \right] = \sum_{k=0}^{m} (n+k) \left[{n+k \atop k} \right]. \tag{6.23}$$

$$\binom{n}{m} = \sum_k \left\{ {n+1 \atop k+1} \right\} \left[{k \atop m} \right] (-1)^{m-k}. \tag{6.24}$$

$$n^{\underline{n-m}} [n \geqslant m] = \sum_k \left[{n+1 \atop k+1} \right] \left\{ {k \atop m} \right\} (-1)^{m-k}. \tag{6.25}$$

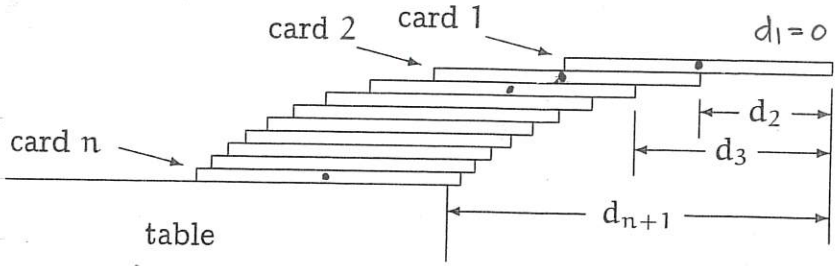
$$\left\{ {n \atop n-m} \right\} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left[{m+k \atop k} \right]. \tag{6.26}$$

$$\left[{n \atop n-m} \right] = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ {m+k \atop k} \right\}. \tag{6.27}$$

$$\left\{ {n \atop l+m} \right\} \binom{l+m}{l} = \sum_k \left\{ {k \atop l} \right\} \left\{ {n-k \atop m} \right\} \binom{n}{k}. \tag{6.28}$$

$$\left[{n \atop l+m} \right] \binom{l+m}{l} = \sum_k \left[{k \atop l} \right] \left[{n-k \atop m} \right] \binom{n}{k}. \tag{6.29}$$

Harmonic numbers



- $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

- **Brick stacking:**

$$\left.\begin{aligned} 1 &\geq d_2 \\ \frac{1+(1+d_2)}{2} &\geq d_3 \\ \frac{1+(1+d_2)+(1+d_3)}{3} &\geq d_4 \\ &\vdots \\ \frac{(1+d_1)+(1+d_2)+\cdots+(1+d_k)}{k} &\geq d_{k+1} \end{aligned}\right\} (1 \leq k \leq n)$$

$$\begin{aligned} k\,d_{k+1} &= d_1 + \cdots + d_k + k \\ (k-1)\,d_k &= d_1 + \cdots + d_{k-1} + k-1 \\ \hline k\,d_{k+1} &= k\,d_k + 1 \end{aligned}$$

$$\begin{cases} d_{k+1} = d_k + \frac{1}{k} \\ d_1 = 0 \end{cases} \Rightarrow d_{k+1} = H_k$$

- **Warm on the rubber band:** reach the other end?

(1) 100 cm, stretch 1 cm: $\frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{100+n-1} \geq 1$?

(2) 〃 , 100 cm: $\frac{1}{100} + \frac{1}{200} + \cdots + \frac{1}{100n} = \frac{1}{100} H_n \geq 1$?

$(16 \leq n < 32)$

- $$H_n = \underbrace{\frac{1}{1}} + \underbrace{\frac{1}{2}+\frac{1}{3}} + \underbrace{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}} + \underbrace{\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}} + \underbrace{\frac{1}{16}+\cdots+\frac{1}{n}}$$

(1) $\frac{\lfloor \lg n \rfloor}{2} + 1 = 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + 8\cdot\frac{1}{16} < H_n < 1 + 2\frac{1}{2} + 4\frac{1}{4} + 8\frac{1}{8} + 16\frac{1}{16} = \lfloor \lg n \rfloor + 1$

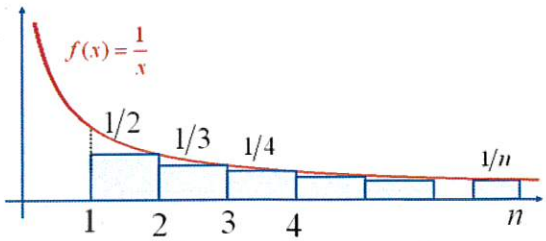
(2) $\ln n + \frac{1}{n} < H_n < \ln n + 1$

(3) $H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right)$

(Euler const) $\gamma = 0.57721\ldots$

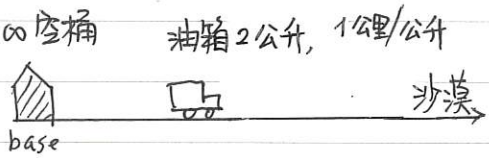
(4) $H_{10^6} = 14.392$ (百萬張卡)

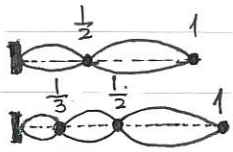
(5) $H_n \geq 100$, $n \geq e^{100-\gamma}$



$$\int_1^n \frac{1}{x}\,dx + \frac{1}{n} < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n$$

- **Jeep problem:** 走多遠?





1 trip : 1

2 trips : $1 + \frac{1}{2}$

3 〃 : $1 + \frac{1}{2} + \frac{1}{3}$ $\quad 2(1-d) \geq 4d \Rightarrow d \leq \frac{1}{3}$

n 〃 : $1 + \frac{1}{2} + \cdots + \frac{1}{n}$, $\quad 2(1-d_n) \geq 2(n-1)\,d_n \Rightarrow d_n \leq \frac{1}{n}$, can travel $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

- $$\sum_{0 \leq k < n} \binom{k}{m} H_k = \sum_0^n \binom{k}{m} H_k \,\Delta k$$

$$\begin{cases} u_k = H_k & \Delta u_k = \frac{1}{k+1} \\ \Delta v_k = \binom{k}{m}, & v_k = \binom{k}{m+1} \end{cases}$$

$$= H_k \binom{k}{m+1}\Big|_0^n - \sum_0^n \binom{k+1}{m+1} \frac{1}{k+1} \Delta k$$

$$= \binom{n}{m+1} H_n - \sum_0^n \frac{1}{m+1} \binom{k}{m} \Delta k$$

$$= \binom{n}{m+1} H_n - \frac{1}{m+1} \binom{k}{m+1}\Big|_0^n = \binom{n}{m+1}\left(H_n - \frac{1}{m+1}\right)$$

$(m=0)$ $\sum_{0 \leq k < n} H_k = n(H_n - 1)$

$(m=1)$ $\sum_{0 \leq k < n} k H_k = \frac{n(n-1)}{2}\left(H_n - \frac{1}{2}\right)$

$\vdots$

Fibonacci numbers

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$$\begin{cases} F_0 = 0; \\ F_1 = 1; \\ F_n = F_{n-1} + F_{n-2}, \quad \text{for } n > 1. \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10	11
F_n	0	1	1	2	3	5	8	13	21	34	55	89

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$$F(z) = F_0 + F_1 z + F_2 z^2 + \cdots = \sum_{n\geq 0} F_n z^n.$$

$$\begin{aligned} F(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + F_5 z^5 + \cdots, \\ zF(z) &= \qquad F_0 z + F_1 z^2 + F_2 z^3 + F_3 z^4 + F_4 z^5 + \cdots, \\ -)\ z^2F(z) &= \qquad\qquad F_0 z^2 + F_1 z^3 + F_2 z^4 + F_3 z^5 + \cdots. \end{aligned}$$

$$F(z) - zF(z) - z^2F(z) = z$$

$$\begin{cases} \alpha+\beta = 1 \\ \alpha\beta = -1 \end{cases} \qquad x^2 - x - 1 = 0$$

$$F(z) = \frac{z}{1-z-z^2} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z} = A\sum_{n\geq 0}(\alpha z)^n + B\sum_{n\geq 0}(\beta z)^n$$

$$= \sum_{n\geq 0}(A\alpha^n + B\beta^n)z^n.$$

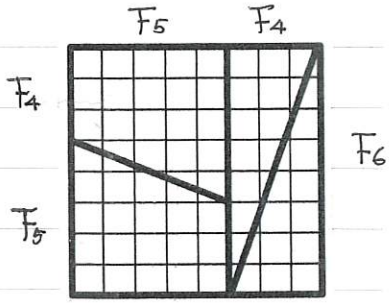
$$\begin{cases} \alpha \\ \beta \end{cases} = \frac{1\pm\sqrt{5}}{2} \qquad \begin{cases} A \\ B \end{cases} = \pm\frac{1}{\sqrt{5}}$$

- $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \approx \frac{1}{\sqrt{5}}\alpha^n \quad (\because |\beta|<1)$ (Binet公式) $\left(\alpha = \frac{1+\sqrt{5}}{2} \approx 1.61803398\right)$ golden ratio

- 定理 (Cassini): $F_{n+1}\cdot F_{n-1} - F_n^2 = (-1)^n$, $\because \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$

- 定理 $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ $\qquad F_5\cdot F_7 - F_6^2 = (-1)^6$

  $\Rightarrow F_n \mid F_{2n},\ F_n \mid F_{3n},\ \ldots,\ F_n \mid F_{kn}$

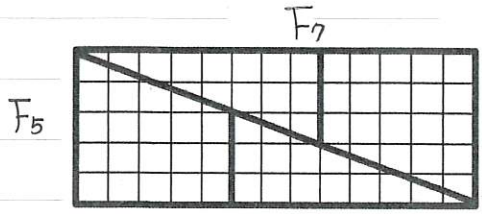


- 定理 $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$

- 定理 (Zeckendorf) $\qquad (a \gg b \Leftrightarrow a \geq b+2)$

  $n = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \quad (k_1 \gg k_2 \gg \cdots \gg k_r \gg 0)$

  $= (b_m b_{m-1} \cdots b_2)_F$ (無連續1)



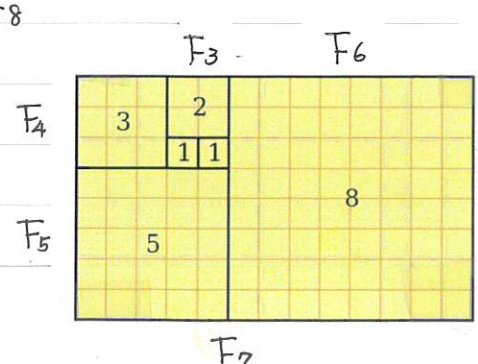
証 $F_{k_1} \leq n < F_{k_1+1} \Rightarrow n = F_{k_1} + m,\ m < F_{k_1-1}$

$F_9 \leq 50 < F_{10} \Rightarrow 50 = F_9 + m,\ m < F_{10} - F_9 = F_8$

$(\because F_8 + F_6 + F_4 + F_2 = F_9 - 1,\ F_9 + F_7 + F_5 + F_3 = F_{10} - 1)$

$50 = F_9 + F_7 + F_4 = 34 + 13 + 3,\quad 51 = (10100101)_F$

$= (10100100)_F \qquad 52 = (10101000)_F$



n	0	1	2	3	4
T_n	1	1	2	3	5

• **Problem** 求 T_n = # ways to pave $2 \times n$ with 1×1

(甲) (Recurrence) $T_n = T_{n-1} + T_{n-2}$ $(n \geq 2)$, $T(z) = \sum_{n \geq 0} T_n z^n$ (Lebesgue Sum) $\qquad T_n = F_{n+1}$

$$T(z) = 1 + zT(z) + z^2 T(z) \Rightarrow T(z) = \frac{1}{1-z-z^2} = \frac{1}{z} F(z)$$

(乙) Symbolic G.F.

$$T = 1 + \square + \square\square + \square\square\square + \square\square\square\square + \square\square\square\square\square + \cdots$$
$$= 1 + \square(1 + \square + \square\square + \square\square\square + \cdots) + \square\square(1 + \square + \square\square + \square\square\square + \cdots)$$
$$= 1 + \square T + \square\square T.$$

$\xrightarrow{\square \to z}$ (Rieman sum) $T(z) = 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + \cdots$

$$= 1 + zT(z) + z^2 T(z)$$

• **Remarks** (1) T_n = # ways to pave $1 \times n$ with 1×1 or 1×2.

(爬 n 階樓梯，每次 1步/2步)

(2) T_n = # Morse codes in n seconds $\begin{cases} \cdot & 1\text{秒} \\ - & 2\text{秒} \end{cases}$

• **Continuants (連分多項式)**

$$K_n(x_1, \ldots, x_n) = K_{n-1}(x_1, \ldots, x_{n-1})x_n + K_{n-2}(x_1, \ldots, x_{n-2}) \quad (n \geq 2)$$
$$K_0() = 1;$$
$$K_1(x_1) = x_1;$$
$$K_2(x_1, x_2) = x_1 x_2 + 1;$$
$$K_3(x_1, x_2, x_3) = x_1 x_2 x_3 + x_1 + x_3;$$
$$K_4(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 + x_1 x_2 + x_1 x_4 + x_3 x_4 + 1$$

ε

$\cdot$

$\cdot\cdot \quad -$

$\cdot\cdot\cdot \quad \cdot- \quad -\cdot$

$\cdot\cdot\cdot\cdot \quad \cdot\cdot- \quad \cdot-\cdot \quad -\cdot\cdot \quad --$

• **定理** (1) $K_n(x_1, \ldots, x_n)$ 含 $K_n(1, 1, \ldots, 1) = F_{n+1}$ 項

(2) $K_n(x_1, \ldots, x_n) \xrightarrow[\text{缺 } x_i x_{i+1} \to -]{x_j \to \cdot} C_n = \{n\text{秒 Morse Codes}\}$, $\quad C_n = C_{n-1}\cdot + C_{n-2}-$

(3) $F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}$ $\quad \begin{cases} k\text{個} & - \\ n-2k\text{個} & \cdot \end{cases}$

(4) $K_n(x_1, \ldots, x_n) = x_1 K_{n-1}(x_2, \ldots, x_n) + K_{n-2}(x_3, \ldots, x_n)$

(5) $\frac{64}{11} = [5, 1, 4, 2] = \frac{K_4(5,1,4,2)}{K_3(1,4,2)} = [5, 1, 4, \underbrace{1, 1}_{1+\frac{1}{1}}] = R^5 L^1 R^4 L^1$

5	64	11	1
	55	9	
4	9	2	2
	8	2	
	1	0	

$$\frac{64}{11} = 5 + \frac{9}{11} = 5 + \frac{1}{\frac{11}{9}}, \quad 64 = 5 \cdot 11 + 9, \quad K_4(5,1,4,2) = 5 \cdot K_3(1,4,2) + K_2(4,2)$$
$$= 5 + \frac{1}{1 + \frac{2}{9}} \qquad 11 = 1 \cdot 9 + 2 \qquad K_3(1,4,2) = 1 \cdot K_2(4,2) + K_1(2)$$
$$= 5 + \frac{1}{1 + \frac{1}{4 + \frac{1}{2}}} \qquad 9 = 4 \cdot 2 + 1 \qquad K_2(4,2) = 4 \cdot K_1(2) + K_0()$$
$$2 = 2 \cdot 1 \qquad K_1(2) = 2$$

Continued Fractions (連分數)

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Date: / /

$$[a_0, a_1, \cdots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}, \quad \frac{64}{11} = 5 + \cfrac{1}{1 + \cfrac{1}{4 + \frac{1}{2}}} = [5, 1, 4, 2]$$

- **定理** $[a_0, a_1, \cdots, a_n]$ 令 $\begin{cases} [a_0, \cdots, a_k] = \dfrac{p_k}{q_k} \\ [a_k, \cdots, a_n] = \dfrac{P_k}{Q_k} \end{cases}$

則 (1) $\begin{cases} p_{-1} = 1, \quad p_0 = a_0, \quad p_k = a_k p_{k-1} + p_{k-2} \\ q_{-1} = 0, \quad q_0 = 1, \quad q_k = a_k q_{k-1} + q_{k-2} \end{cases}$ $(1 \le k \le n)$, 且 $\begin{vmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{vmatrix} = (-1)^k$ $(p_k \perp q_k)$

k	-1	0	1	2	3
a_k		5	1	4	2
p_k	1	5	6	29	64
q_k	0	1	1	5	11

(2) $\begin{cases} P_{n+1} = 1, \quad P_n = a_n, \quad P_k = a_k P_{k+1} + P_{k+2} \\ Q_n = 1, \quad Q_k = P_{k+1} \end{cases}$ $(0 \le k \le n-1)$

k	0	1	2	3	4
a_k	5	1	4	2	
P_k	64	11	9	2	1
Q_k	11	9	2	1	

(3) $[a_0, a_1, \cdots, a_n] = \dfrac{p_n}{q_n} = \dfrac{P_0}{Q_0} = \dfrac{P_0}{P_1} = \dfrac{K_{n+1}(a_0, \cdots, a_n)}{K_n(a_1, \cdots, a_n)}$

証明 (Induction)

(1) $[a_0] = \dfrac{a_0}{1}$, $[a_0, a_1] = a_0 + \dfrac{1}{a_1} = \dfrac{a_1 a_0 + 1}{a_1}$, $\dfrac{p_{k+1}}{q_{k+1}} = [a_0, \cdots, a_k, a_{k+1}] = [a_0, \cdots, a_k + \dfrac{1}{a_{k+1}}]$

$$= \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}} = \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}}$$

$$= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}}$$

$$\begin{cases} \begin{vmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{vmatrix} = \begin{vmatrix} 1 & a_0 \\ 0 & 1 \end{vmatrix} = (-1)^0 \\ \begin{vmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{vmatrix} = \begin{vmatrix} p_k & a_{k+1} p_k + p_{k-1} \\ q_k & a_{k+1} q_k + q_{k-1} \end{vmatrix} = - \begin{vmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{vmatrix} \end{cases}$$

(2) $[a_n] = \dfrac{a_n}{1}$, $[a_{n-1}, a_n] = \dfrac{a_{n-1} a_n + 1}{a_n}$, $\dfrac{P_{k-1}}{Q_{k-1}} = [a_{k-1}, a_k, \cdots, a_n] = a_{k-1} + \cfrac{1}{\frac{P_k}{Q_k}} = \dfrac{a_{k-1} P_k + P_{k+1}}{P_k}$

- **Remark**

解整數方程式: $192x + 33y = 15$

$(192, 33) = 3 \mid 15$, $64x + 11y = 5$, $64 \cdot 5 - 11 \cdot 29 = 1$

$64 \cdot 25 - 11 \cdot 145 = 5$

• **Example 1** $\sqrt{2} = 1.41421356\ldots$

(1) $\sqrt{2} = 1 + x \quad (\because 1 < \sqrt{2} < 2), \quad x = \sqrt{2} - 1, \quad x^2 + 2x = 1$

(2) $x = \dfrac{1}{2+x}$

(3) $\sqrt{2} = 1 + x = 1 + \dfrac{1}{2+x} = 1 + \dfrac{1}{2 + \dfrac{1}{2+x}} = \cdots = 1 + \dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2 + \cdots}}}}$

$= [1, 2, 2, 2, 2, 2, \ldots]$

(4) $[1, 2] = \frac{3}{2} = 1.5$

$[1, 2, 2] = \frac{7}{5} = 1.4$

$[1, 2, 2, 2] = \frac{17}{12} = 1.41666666$

$[1, 2, 2, 2, 2] = \frac{41}{29} = 1.41379310$

$[1, 2, 2, 2, 2, 2] = \frac{99}{70} = 1.41428571$

$[1, 2, 2, 2, 2, 2, 2] = \frac{239}{169} = 1.41420118$

k	-1	0	1	2	3	4	5	6	7	$\cdots$
a_k		1	2	2	2	2	2	2	2	$\cdots$
p_k	1	1	3	7	17	41	99	239	577	$\cdots$
q_k	0	1	2	5	12	29	70	169	408	$\cdots$

• **Example 2** (Golden ratio) $\alpha = \dfrac{1+\sqrt{5}}{2} = 1.61803398\ldots$

$\left(F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \right.$
$\left. \alpha, \beta : x^2 - x - 1 = 0 \text{ 之根} \right)$

(A) $\alpha = 1 + \dfrac{1}{\alpha} = 1 + \dfrac{1}{1 + \frac{1}{\alpha}} = 1 + \dfrac{1}{1 + \dfrac{1}{1 + \frac{1}{\alpha}}} = \cdots = 1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1 + \cdots}}}}$

$= [1, 1, 1, 1, 1, \ldots]$

$\alpha \approx [1, 1, 1, 1, 1, 1, 1] = \frac{21}{13} = 1.61538461\ldots$

$\approx [1, 1, 1, 1, 1, 1, 1, 1] = \frac{34}{21} = 1.61904761\ldots$

k	-1	0	1	2	3	4	5	6	7	$\cdots$	k
a_k		1	1	1	1	1	1	1	1	$\cdots$	1
p_k	1	1	2	3	5	8	13	21	34	$\cdots$	F_{k+2}
q_k	0	1	1	2	3	5	8	13	21	$\cdots$	F_{k+1}

(B) $\alpha = \sqrt{1+\alpha} = \sqrt{1+\sqrt{1+\alpha}} = \sqrt{1+\sqrt{1+\sqrt{1+\alpha}}} = \cdots = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$