Generating Functions

\[
\begin{align*}
\langle q_n \rangle \to G(z) &= \sum_{n=0}^{\infty} q_n z^n = q_0 + q_1 z + \cdots + q_n z^n + \cdots \quad q_n = [z^n] G(z) \\
\langle f_n \rangle \to F(z) &= \sum_{n=0}^{\infty} f_n z^n = f_0 + f_1 z + \cdots + f_n z^n + \cdots
\end{align*}
\]

Table 334 Generating function manipulations.

\[
\alpha F(z) + \beta G(z) = \sum_n (\alpha f_n + \beta g_n) z^n
\]

\[
z^m G(z) = \sum_n g_{n-m} z^n, \quad (\text{integer } m \geq 0) = g_0 z^m + g_1 z^{m+1} + \cdots
\]

\[
\frac{G(z) - g_0 - g_1 z - \cdots - g_{m-1} z^{m-1}}{z^m} = \sum_{n \geq 0} g_{n+m} z^n, \quad (\text{integer } m \geq 0) = g_m + g_{m+1} z + \cdots
\]

\[
G(cz) = \sum_n c^n g_n z^n
\]

\[
G'(z) = \sum_n (n+1) g_{n+1} z^n
\]

\[
z G'(z) = \sum_n n g_n z^n
\]

\[
\int_0^z G(t) \, dt = \sum_{n \geq 1} \frac{1}{n} g_{n-1} z^n
\]

\[
F(z) G(z) = \sum_n \left( \sum_k f_k g_{n-k} \right) z^n = \sum_{k+\ell=n} f_k g_{\ell}
\]

\[
\left( 1 + z + z^2 + \cdots + z^n + \cdots \right) G(z) = \frac{1}{1-z} G(z) = \sum_n \left( \sum_{k \leq n} g_k \right) z^n
\]

\[
\begin{align*}
\frac{G(z) - G(0)}{z} &= g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots \\
\frac{G(z) - G(0)}{z} &= g_0 + g_1 z + g_2 z^2 + g_3 z^3 + g_4 z^4 + \cdots
\end{align*}
\]

\[
\sum_n f_{2n} z^n = \frac{1}{2} \left( \frac{z}{1-z-z^2} + \frac{-z}{1+z-z^2} \right) = \frac{z^2}{1-3z+z^2}
\]

\[
\sum_n f_{2n} z^n = \frac{z}{1-3z+z^2}
\]
<table>
<thead>
<tr>
<th>sequence</th>
<th>generating function</th>
<th>closed form</th>
</tr>
</thead>
<tbody>
<tr>
<td>⟨1, 0, 0, 0, 0, 0, 0, . . .⟩</td>
<td>$\sum_{n \geq 0} [n = 0] z^n$</td>
<td>1</td>
</tr>
<tr>
<td>⟨0, . . ., 0, 1, 0, 0, 0, . . .⟩</td>
<td>$\sum_{n \geq 0} [n = m] z^n$</td>
<td>$z^m$</td>
</tr>
<tr>
<td>⟨1, 1, 1, 1, 1, 1, 1, . . .⟩</td>
<td>$\sum_{n \geq 0} z^n$</td>
<td>$\frac{1}{1-z}$</td>
</tr>
<tr>
<td>⟨1, −1, 1, −1, 1, −1, 1, . . .⟩</td>
<td>$\sum_{n \geq 0} (-1)^n z^n$</td>
<td>$\frac{1}{1+z}$</td>
</tr>
<tr>
<td>⟨1, 0, 1, 0, 1, 0, 1, . . .⟩</td>
<td>$\sum_{n \geq 0} [2\backslash n] z^n$</td>
<td>$\frac{1}{1-z^2}$</td>
</tr>
<tr>
<td>⟨1, 0, . . ., 0, 1, 0, . . ., 0, 1, 0, . . .⟩</td>
<td>$\sum_{n \geq 0} [m\backslash n] z^n$</td>
<td>$\frac{1}{1-z^m}$</td>
</tr>
<tr>
<td>⟨1, 2, 3, 4, 5, 6, . . .⟩</td>
<td>$\sum_{n \geq 0} (n+1) z^n$</td>
<td>$\frac{1}{(1-z)^2}$</td>
</tr>
<tr>
<td>⟨1, 2, 4, 8, 16, 32, . . .⟩</td>
<td>$\sum_{n \geq 0} 2^n z^n$</td>
<td>$\frac{1}{1-2z}$</td>
</tr>
<tr>
<td>⟨1, 4, 6, 4, 1, 0, 0, . . .⟩</td>
<td>$\sum_{n \geq 0} \binom{4}{n} z^n$</td>
<td>$(1+z)^4$</td>
</tr>
<tr>
<td>⟨1, c, (c/2), (c/3), . . .⟩</td>
<td>$\sum_{n \geq 0} \binom{c}{n} z^n$</td>
<td>$(1+z)^c$</td>
</tr>
<tr>
<td>⟨1, c, (c+1)/2, (c+2)/3, . . .⟩</td>
<td>$\sum_{n \geq 0} \binom{c+n-1}{n} z^n$</td>
<td>$\frac{1}{(1-z)^c}$</td>
</tr>
<tr>
<td>⟨1, c, c^2, c^3, . . .⟩</td>
<td>$\sum_{n \geq 0} c^n z^n$</td>
<td>$\frac{1}{1-cz}$</td>
</tr>
<tr>
<td>⟨1, (m+1)/m, (m+2)/m, (m+3)/m, . . .⟩</td>
<td>$\sum_{n \geq 0} \binom{m+n}{m} z^n$</td>
<td>$\frac{1}{(1-z)^{m+1}}$</td>
</tr>
<tr>
<td>⟨0, 1, 1/2, 1/3, 1/4, . . .⟩</td>
<td>$\sum_{n \geq 1} \frac{1}{n} z^n$</td>
<td>$\ln \frac{1}{1-z}$</td>
</tr>
<tr>
<td>⟨0, 1, −1/2+1/3, −1/4, . . .⟩</td>
<td>$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} z^n$</td>
<td>$\ln(1+z)$</td>
</tr>
<tr>
<td>⟨1, 1/2, 1/6, 1/24, 1/120, . . .⟩</td>
<td>$\sum_{n \geq 0} \frac{1}{n!} z^n$</td>
<td>$e^z$</td>
</tr>
</tbody>
</table>
Table 351 Generating functions for special numbers.

\[ \frac{1}{(1-z)^{m+1}} \ln \frac{1}{1-z} = \sum_{n \geq 0} (H_{m+n} - H_m) \binom{m+n}{n} z^n \]  
(7.43)

\[ \frac{z}{e^z - 1} = \sum_{n \geq 0} B_n \frac{z^n}{n!} \]  
(7.44)

\[ \frac{F_m z}{1 - (F_{m-1} + F_{m+1})z + (-1)^m z^2} = \sum_{n \geq 0} F_{mn} z^n \]  
(7.45)

\[ \sum_{k} \left\{ \binom{m}{k} \frac{k! z^k}{(1-z)^{k+1}} \right\} = \sum_{n \geq 0} n^m z^n \]  
(7.46)

\[ (z^{-1})^{-m} = \frac{z^m}{(1-z)(1-2z) \cdots (1-mz)} = \sum_{n \geq 0} \binom{n}{m} z^n \]  
(7.47)

\[ z^m = z(z+1) \cdots (z+m-1) = \sum_{n \geq 0} \binom{m}{n} z^n \]  
(7.48)

\[ (e^z - 1)^m = m! \sum_{n \geq 0} \binom{n}{m} \frac{z^n}{n!} \]  
(7.49)

\[ (\ln \frac{1}{1-z})^m = m! \sum_{n \geq 0} \binom{n}{m} \frac{z^n}{n!} \]  
(7.50)

\[ \left( \frac{z}{\ln(1+z)} \right)^m = \sum_{n \geq 0} \frac{z^n}{n!} \binom{m}{m-n} \binom{m-1}{n} \]  
(7.51)

\[ \left( \frac{z}{1-e^{-z}} \right)^m = \sum_{n \geq 0} \frac{z^n}{n!} \binom{m}{m-n} \binom{m-1}{n} \]  
(7.52)

\[ e^{z+wz} = \sum_{m,n \geq 0} \binom{n}{m} w^m \frac{z^n}{n!} \]  
(7.53)

\[ e^{w(e^z-1)} = \sum_{m,n \geq 0} \binom{n}{m} w^m \frac{z^n}{n!} \]  
(7.54)

\[ \frac{1}{(1-z)^w} = \sum_{m,n \geq 0} \binom{n}{m} w^m \frac{z^n}{n!} \]  
(7.55)

\[ \frac{1-w}{e^{(w-1)z} - w} = \sum_{m,n \geq 0} \binom{n}{m} w^m \frac{z^n}{n!} \]  
(7.56)
Solve problem (1) recurrence of $g_n + 5$ term
(2) $G(z)$ equation
(3) solve $G(z)$
(4) expand $G(z)$, get $g_n$

Example 1 (Fibonacci)

\[
\begin{align*}
g_0 &= 0; \quad g_1 = 1; \\
g_n &= g_{n-1} + g_{n-2}, \quad \text{for } n \geq 2.
\end{align*}
\]

\[
\begin{align*}
G(z) &= \sum_n g_n z^n \\
&= \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n z^n \\
&= z G(z) + z^2 G(z) + z
\end{align*}
\]

Example 3 (Mutual recurrences)

\[
\begin{align*}
&U_m = \text{# ways to pave } \begin{array}{c}
\ldots \\
\ldots \\
\end{array} \\
&V_m = \begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
U_0 &= 1, \quad U_1 = 0; \quad V_0 = 0, \quad V_1 = 1; \\
U_n &= 2V_{n-1} + U_{n-2}, \quad V_n = U_{n-1} + V_{n-2}, \quad \text{(for } n \geq 2) \\
\Rightarrow U_n &= 2V_{n-1} + U_{n-2} + [n=0], \quad V_n = U_{n-1} + V_{n-2}, \quad \forall n
\end{align*}
\]

\[
\begin{align*}
U(z) &= 2zV(z) + z^2U(z) + 1, \\
V(z) &= zU(z) + z^2V(z)
\end{align*}
\]

\[
\begin{align*}
U(z) &= \frac{1 - z^2}{1 - 4z^2 + z^4}; \\
V(z) &= \frac{z}{1 - 4z^2 + z^4}.
\end{align*}
\]

\[
\begin{align*}
U_{n+1} &= W_n = \frac{3 + 2\sqrt{3}}{6}(2 + \sqrt{3})^n + \frac{3 - 2\sqrt{3}}{6}(2 - \sqrt{3})^n; \\
\text{then } \sum_{n=0}^\infty W_n z^n = \frac{1}{1 - 4z + z^2}
\end{align*}
\]

\[
\begin{align*}
V_{n+1} &= W_n - W_{n-1} = \frac{3 + \sqrt{3}}{6}(2 + \sqrt{3})^n + \frac{3 - \sqrt{3}}{6}(2 - \sqrt{3})^n \\
&= \frac{(2 + \sqrt{3})^n}{3 - \sqrt{3}} + \frac{(2 - \sqrt{3})^n}{3 + \sqrt{3}}.
\end{align*}
\]
Example 4  How many ways to pay 50z with?

\[
\begin{align*}
P &= z + z^2 + z^3 + z^4 + \ldots, \\
N &= (z + z^2 + z^3 + z^4 + \ldots)P, \\
D &= (z + z^2 + z^3 + z^4 + \ldots)N, \\
Q &= (z + z^2 + z^3 + z^4 + \ldots)D, \\
C &= (z + z^2 + z^3 + z^4 + \ldots)Q.
\end{align*}
\]

\[
\begin{align*}
(1 - z) P &= 1, \\
(1 - z^5) N &= P, \\
(1 - z^{10}) D &= N, \\
(1 - z^{25}) Q &= D, \\
(1 - z^{50}) C &= Q.
\end{align*}
\]

\[
\begin{align*}
P_n &= P_{n-1} + [n = 0], \\
N_n &= N_{n-5} + P_n, \\
D_n &= D_{n-10} + N_n, \\
Q_n &= Q_{n-25} + D_n, \\
C_n &= C_{n-50} + Q_n.
\end{align*}
\]

\[
\begin{array}{c|cccccccccc}
\hline
\text{ } & 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 \\
\hline
P_n & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
N_n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
D_n & 1 & 2 & 4 & 6 & 9 & 12 & 16 & 25 & 36 & 50 & 49 \\
Q_n & 1 & 1 & 13 & 49 & 50 & 1 & 1 & 1 & 1 & 1 & 1 \\
C_n & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

(1) \( C_n = ? \)

\[
\begin{align*}
C(z) &= \sum_{n = 0}^{\infty} C_n z^n = \left\{ \begin{array}{l}
1 + z + z^2 + \ldots \\
1 + z^5 + z^{10} + z^{15} + \ldots \\
1 + z^{10} + z^{20} + z^{30} + \ldots \\
1 + z^{25} + z^{50} + z^{100} + \ldots \\
1 + z^{50} + z^{100} + z^{150} + z^{200} + \ldots
\end{array} \right. \\
&= \frac{1}{(1 - z)(1 - z^5)(1 - z^{10})(1 - z^{25})(1 - z^{50})}.
\end{align*}
\]

\[
G(z) = \sum_{k = 0}^{\infty} a_k z^k = \frac{4}{(1 - z)(1 - z^5)(1 - z^{10})(1 - z^{25})(1 - z^{50})},
\]

\[
\begin{align*}
G(z) &= \frac{(1 + z + z^2 + \ldots + z^9)^2 (1 + z^5 + z^{10} + z^{15} + \ldots (1 + z^{25} + z^{50} + z^{100} + \ldots (1 + z^{50} + z^{100} + z^{150} + z^{200} + \ldots } \\
&= \frac{(1 + z^2 + \ldots + z^9)^2 (1 + z^5 + z^{10} + \ldots (1 + z^{25} + z^{50} + \ldots (1 + z^{50} + z^{100} + \ldots } \\
&= \frac{(1 + z + z^2 + \ldots + z^9)^2 (1 + z^5 + z^{10} + z^{15} + \ldots (1 + z^{25} + z^{50} + z^{100} + \ldots (1 + z^{50} + z^{100} + z^{150} + z^{200} + \ldots
\end{align*}
\]

(2) \( a_k = ? \)

\[
\begin{align*}
C_{50} &= a_{10} \quad = A_0 \left( \frac{5}{4} \right) + A_0 \left( \frac{4}{4} \right) = 50, \\
C_{100} &= a_{20} \quad = A_0 \left( \frac{2}{4} \right) + A_0 \left( \frac{4}{4} \right) + A_{20} \left( \frac{4}{4} \right) = 292, \\
C_{400} &= a_{40} \quad = A_0 \left( \frac{4}{4} \right) + A_0 \left( \frac{4}{4} \right) + A_{40} \left( \frac{4}{4} \right) = 2940, \\
C_{500} &= a_{100} = A_0 \left( \frac{4}{4} \right) + A_0 \left( \frac{4}{4} \right) + A_{10} \left( \frac{4}{4} \right) + A_{100} \left( \frac{4}{4} \right) + A_{20} \left( \frac{4}{4} \right) = 2910, \\
A_{10} &= A_0 \left( \frac{4}{4} \right) + A_0 \left( \frac{4}{4} \right) + A_{10} \left( \frac{4}{4} \right) + A_{10} \left( \frac{4}{4} \right) + A_{20} \left( \frac{4}{4} \right) + A_{20} \left( \frac{4}{4} \right) + A_{30} \left( \frac{4}{4} \right) + A_{30} \left( \frac{4}{4} \right) + A_{40} \left( \frac{4}{4} \right) + A_{40} \left( \frac{4}{4} \right) + A_{50} \left( \frac{4}{4} \right) = 2910.
\end{align*}
\]

(0 ≤ y < 10)
Example 6 \( f_n = \#\) spanning trees of \( n \)-fan.

解 (甲)

\[
\begin{align*}
  f_1 &= 1 \\
  f_n &= f_{n-1} + f_{n-2} + \ldots + f_1 + 1 \quad (n \geq 2) \\
  \Rightarrow f_n &= f_{n-1} + \sum_{K<n} f_k + \delta_{n=1}, \quad \forall n
\end{align*}
\]

\[
T(z) = z \sum f(z) + (z + z^2 + \ldots) F(z) + (z + z^2 + \ldots)
\]

\[
= z \frac{z}{1-z} T(z) + \frac{z}{1-z}
\]

\[
T(z) = \frac{z}{1-3z + z^2}
\]

\[
\therefore \ f_n = F_{2n}
\]

\( f_3 = 8 \)

\[
\begin{align*}
  f_4 &= 4 + 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 = 21.
\end{align*}
\]

\( f_n = \sum_{m>0} \sum_{k_1+k_2+\ldots+k_m=n} k_1k_2\ldots k_m \)

\[
G(z) = \sum_{m>0} \left( \sum_{k_1+k_2+\ldots+k_m=n} \frac{k_1k_2\ldots k_m}{k_1+\ldots+k_m} \right) z^m
\]

\[
= \sum_{m>0} \sum_{k_1+\ldots+k_m=n} \frac{k_1k_2\ldots k_m}{k_1+\ldots+k_m} z^m
\]

\[
= \sum \left( \sum_{k_1>0} \frac{k_1}{k_1} \right)^m \left( \sum_{k_2>0} \frac{k_2}{k_2} \right) \ldots \left( \sum_{k_m>0} \frac{k_m}{k_m} \right)
\]

\[
= \sum_{m>0} G(z)^m
\]

\[
= \frac{G(z)}{1-G(z)} = \frac{z}{1-3z + z^2}
\]

\[
G(z) = \sum \frac{k}{z^k} = z + 2z^2 + 3z^3 + \ldots = \frac{z}{(1-z)^2}
\]
Catalan numbers

(1) \( C_n = \# \) binary trees with \( n \) nodes
\[ C_n = C_{n-1} + C_{n-2} + C_{n-3} + \cdots + C_0 \]
\[ = \sum_{k=0}^{n} C_k C_{n-k} \]
\[ C(\z) = \sum_{n=0}^{\infty} C_n \z^n = \frac{1}{2} \left[ 1 - \frac{1 - 4\z}{\sqrt{1 - 4\z}} \right] \]
\[ C_n = \frac{1}{n+1} \frac{(-1)^n}{2^n} \left[ \frac{1}{n+1} \right] \left( \frac{2n}{n+1} \right) \]

(2) \( C_n = \# \) ways to multiply (parenthesize) \( x_0 \cdot x_1 \cdot \ldots \cdot x_n \)
\[ (x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)), \quad (x_0 \cdot (x_1 \cdot x_2) \cdot x_3), \quad ((x_0 \cdot x_1) \cdot (x_2 \cdot x_3)), \quad ((x_0 \cdot x_1) \cdot x_2 \cdot x_3) \]
\[ C_3 = 5 \]

(3) \( C_n = \# \) ways to triangulate \( \{x_0, x_1, \ldots, x_n\} \)

(4) \( C_n = \# \) sequences \( (a_1, a_2, \ldots, a_{2n}) \) \( \{a_1 + a_2 + \cdots + a_{2n} = 0\} \)
\[ a_i + a_{i+1} + a_{i+2} \geq 0, \quad 1 \leq i \leq 2n \]

\( A = \{+1, \rightarrow, (, \text{ push }, 1, \rightarrow, ), \text{ pop }, 0 \} \)

(5) \( 110010001110 \rightarrow (n, m) \)
\[ 10001 \rightarrow (n+1, m-1) \]

(6) \( C_n = \frac{1}{n+1} \frac{2n}{n+1} \)

(7) \( C_n = \frac{2n}{n+1} \frac{2n}{n+1} \)

\[ C_n = \frac{2n}{n+1} \frac{2n}{n+1} \]

\[ C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14 \]
\[ T(z) = 1 + z T(z)^2 \]

\[ R = 1 + z (1 + z + z^2 + z^2 + \ldots) (1 + z + z^2 + z^2 + \ldots) \]

\[ \mathcal{J} = \{ t \mid t: \text{Binary tree} \} \quad |t| = \begin{cases} \# \text{nodes of } t, \\ 0 \end{cases}, \quad t = \emptyset \]

\[ T(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{t \in \mathcal{J}} |t| z^{|t|} = 1 + \sum_{t \in \mathcal{J}} z^{|t_l| + |t_r|} = 1 + z \sum_{t \in \mathcal{J}} \sum_{t_l, t_r \in \mathcal{J}} z^{|t_l|} z^{|t_r|} = 1 + z T(z)^2 \]

**Symbolic method**

1. \( \mathcal{C} \) : class of combinatorial objects
   - Example: \( \mathcal{J} \)
2. \( |w|_\mathcal{C} \) : size of \( w \in \mathcal{C} \), \( C_n = \# \text{of size } n \).
   - \( |t| = \# \text{nodes of } t \in \mathcal{J} \)
3. Structural relation of \( \mathcal{C} \)
4. Equation of \( C(z) = \sum_{w \in \mathcal{C}} z^{|w|} = \sum_{n=0}^{\infty} C_n z^n \) \( \Rightarrow \) \( T(z) = 1 + T(z) \times T(z) \)

**Lemma**

1. (disjoint union) \( \mathcal{C} = A + B \), \( |w|_\mathcal{C} = \begin{cases} |w|_A, & w \in A \\ |w|_B, & w \in B \end{cases} \Rightarrow C(z) = A(z) + B(z) \)
2. (Cartesian product) \( \mathcal{C} = A \times B \), \( |(a, b)|_\mathcal{C} = |a|_A \times |b|_B \Rightarrow C(z) = A(z) B(z) \)
3. (Sequence) \( \mathcal{C} = A^* = \{ \varepsilon \} + A + A \times A + A \times A \times A + \ldots \Rightarrow C(z) = \frac{1}{1 - A(z)} \)

**Proof**

1. \( C(z) = \sum_{w \in \mathcal{C}} z^{|w|} = \sum_{w \in A} z^{|w|} + \sum_{w \in B} z^{|w|} = A(z) + B(z) \) \( (C_n = C_n + C_n) \)
2. \( C(z) = \sum_{(a, b) \in A \times B} z^{|(a, b)|} = \sum_{a \in A, b \in B} z^{|a|} z^{|b|} = \sum_{a \in A} z^{|a|} \sum_{b \in B} z^{|b|} = A(z) B(z) \) \( (C_n = \sum_{k=0}^{n} C_k b_k) \)
3. \( C(z) = 1 + A(z) + A^2(z) + A^3(z) + \ldots \)
**Examples**

1. Binary strings: \( b = (0+1)^* \)
   \(|b| = \# \text{ bits in } b\)
   \( B(z) = \frac{1}{1-2z} \)
   \( b_n = 2^n \)

2. Binary strings: \( B = (0+1)(0+1)B \)
   \( B(z) = 1 + 2zB(z) \)

3. Binary strings: \( \hat{B} = (0+1)^* \)
   \( \hat{B}(z) = 1 + z + (x^2 + x)B(z) \)

4. Binary trees: \( J = \emptyset + J \times J \)
   \( T(z) = 1 + zT(z)T(z) \)
   \( C_n = \frac{2n}{n+1} \binom{2n}{n} \)

5. Binary trees: \( \bigodot J = \emptyset + J \times J \)
   \( \bigotimes T(z) = z + T(z)T(z) \)
   \( C_n = C_{n-1} = \frac{(2n-2)}{n} \binom{2n-2}{n-1} \)

6. Non-empty General trees: \( G = \{\} \times G^* \)
   \( G(z) = \frac{2z}{1-G(z)} \)
   \( G_n = C_{n+1} = \frac{2n-2}{n} \binom{2n-2}{n-1} \)

7. Floor tiling: \( T = 1 + 0 + 0 + 0 + 0 + 0 + 0 + \cdots \)
   \( = 1 + 0(T + 0 + 0 + 0 + \cdots) + 0(T + 0 + 0 + 0 + \cdots) \)
   \( T(z) = 1 + zT(z) + z^2T(z) \)
   \( T_n = F_{n+1} \)
**Ordinary G.F.**

\[
\begin{align*}
A(z) &= \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots \\
B(z) &= \sum_{k=0}^{\infty} b_k z^k = b_0 + b_1 z + b_2 z^2 + \cdots + b_k z^k + \cdots \\
A(z) B(z) &= C(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_n = \sum_{0 \leq k \leq n} a_k b_{n-k} \\
C_0 &= a_0 b_0 + a_1 b_1 + a_2 b_2 + \cdots + a_k b_0
\end{align*}
\] (Convolution)

**Exponential G.F.**

\[
\begin{align*}
\hat{A}(z) &= \sum_{k=0}^{\infty} a_k \frac{z^k}{k!} = a_0 + \frac{a_1 z}{1!} + \frac{a_2 z^2}{2!} + \cdots + \frac{a_k z^k}{k!} + \cdots \\
\hat{B}(z) &= \sum_{k=0}^{\infty} b_k \frac{z^k}{k!} = b_0 + \frac{b_1 z}{1!} + \frac{b_2 z^2}{2!} + \cdots + \frac{b_k z^k}{k!} + \cdots \\
\hat{A}(z) \hat{B}(z) &= \hat{C}(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}, \quad c_n = \sum_{0 \leq k \leq n} \binom{n}{k} a_k b_{n-k} \\
C_0 &= a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_k b_0 \\
C_1 &= a_0 b_1 + a_1 b_2 + a_2 b_3 + \cdots + a_k b_1
\end{align*}
\]

**Dirichlet G.F.**

\[
\begin{align*}
\tilde{A}(z) &= \sum_{k=1}^{\infty} \frac{a_k}{k^s} = a_1 \frac{1}{1^s} + a_2 \frac{2}{2^s} + a_3 \frac{3}{3^s} + \cdots + \frac{a_k}{k^s} + \cdots \\
\tilde{B}(z) &= \sum_{k=1}^{\infty} \frac{b_k}{k^s} = b_1 \frac{1}{1^s} + b_2 \frac{2}{2^s} + b_3 \frac{3}{3^s} + \cdots + \frac{b_k}{k^s} + \cdots \\
\tilde{A}(z) \tilde{B}(z) &= \tilde{C}(z) = \sum_{m=1}^{\infty} \frac{c_m}{m^s}, \quad c_m = \sum_{d \mid m} a_d b_m \mu\left(\frac{m}{d}\right) = \sum_{d \mid m} a_d b_k \\
C_0 &= a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_k b_1
\end{align*}
\] (C = a * b)

**定理 (1) \( \zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \frac{1}{1 - 1^z} \cdot \frac{1}{1 - 2^z} \cdot \frac{1}{1 - 3^z} \cdot \frac{1}{1 - 4^z} \cdots \) (Riemann Zeta fn)

\[
= \left(1 + \frac{1}{2^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots \right) \left(1 + \frac{1}{3^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots \right) \left(1 + \frac{1}{5^z} + \frac{1}{3^z} + \frac{1}{5^z} + \cdots \right) \cdots
\]

**定理 (2) \( \mu(z) = \frac{\mu(1)}{1^z} + \frac{\mu(2)}{2^z} + \frac{\mu(3)}{3^z} + \frac{\mu(4)}{4^z} + \cdots = \left(1 - \frac{1}{2^z}\right) \left(1 - \frac{1}{3^z}\right) \left(1 - \frac{1}{5^z}\right) \cdots \)

**定理 (3) \( \zeta(z) \mu(z) = 1 \)

**定理 (1) \( G(z) = \zeta(z) F(z) \Leftrightarrow F(z) = \mu(z) G(z) \)

(2) \( g * f \Leftrightarrow f = \mu * g \)

(3) \( g_m = \sum_{d \mid m} f_d \Leftrightarrow f_m = \sum_{d \mid m} \mu(d) g_m \) (Möbius inversion)