Discrete Probability

- **統計**：描述過去

$N = 50 \qquad f_k = (\# x_i = k)$

$x_1 = 30$, $x_2 = 50$, $x_i = 10$, $x_{50} = 30$

$f_0 = 2$, $f_{10} = 3$, $f_{30} = 15$, $\vdots$, $f_{100} = 1$

- **機率**：預測未來

$p_k = \frac{f_k}{N} = P(X = k)$

- 平均值：$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i = \sum_{k=0}^{100} \frac{f_k}{N} k = \sum_{k=0}^{100} p_k k$ $= E(X)$：期望值
- 標準差²：$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2 = \sum_{k=0}^{100}\frac{f_k}{N}(k-\mu)^2 = \sum_{k=0}^{100} p_k (k-\mu)^2 = E((X-\mu)^2)$ $= Var(X)$：變異數

$$= \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \mu^2 = \sum_k \frac{f_k}{N} k^2 - \mu^2 = \sum_k p_k k^2 - \mu^2 = E(X^2) - \mu^2$$

- Markov 不等式：$P(X \geq c\mu) \leq \frac{1}{c}$

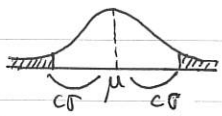
Chebyshev 不等式：$P((X-\mu)^2 \geq c^2\sigma^2) \leq \frac{1}{c^2}$

$P(|X-\mu| \geq c\sigma) \leq \frac{1}{c^2}$

$\mu = 30$分, 50人 (總分 = 1500)

≥ 60分, ≤ 25人, ≥ 90分, $\leq \frac{50}{3}$人

$\geq c\mu$, $\leq \frac{N}{c}$



Probability Space (Ω, P, X)

- Sample space: $\Omega = \{\omega_1, \omega_2, \cdots\}$
- Probability: $P: \Omega \to [0,1]$, $\sum_{\omega\in\Omega} P(\omega) = 1$
- Random variable: $X: \Omega \to \mathbb{R}$,

Ω	⚀	⚁	⚂	⚃	⚄	⚅
P	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
X	1	2	3	4	5	6
Y	5	-2	5	-2	5	-2
X^2	1	4	9	16	25	36

- $E(X) = \sum_{\omega\in\Omega} X(\omega)P(\omega) = \sum_{k\in X(\Omega)} k\,P(X=k) = \mu_X$

$Var(X) = E((X-\mu)^2) = E(X^2) - \mu^2 = \sigma_X^2$

$E(X) = 1\frac{1}{6} + 2\frac{1}{4} + 3\frac{1}{12} + 4\frac{1}{4} + 5\frac{1}{6} + 6\frac{1}{12} = \frac{39}{12}$

$E(Y) = 5\frac{1}{6} - 2\frac{1}{4} + 5\frac{1}{12} - 2\frac{1}{4} + 5\frac{1}{6} - 2\frac{1}{12} = \frac{11}{12}$

$= 5\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{6}\right) - 2\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{12}\right)$

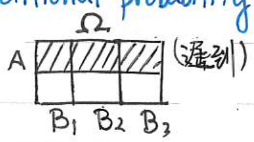
- Event: $A \subset \Omega$, $P(A) = \sum_{\omega\in A} P(\omega)$

$P(X=a) = P(\{\omega\in\Omega \mid X(\omega) = a\}) = P(X^{-1}(a))$

- Conditional probability

$P(A|B) = \frac{P(A\cap B)}{P(B)}$ $\quad \begin{cases} A = X^{-1}(a) \\ B = Y^{-1}(b) \end{cases}$

$P(X=a|Y=b) = \frac{P(X=a, Y=b)}{P(Y=b)}$



$Var(X) = \left(1-\frac{39}{12}\right)^2\frac{1}{6} + \cdots + \left(6-\frac{39}{12}\right)^2\frac{1}{12} = \frac{363}{144}$

$= \left(1^2\frac{1}{6} + 2^2\frac{1}{4} + \cdots + 6^2\frac{1}{12}\right) - \left(\frac{39}{12}\right)^2 = 〃$

$G_X(z) = \frac{1}{6}z + \frac{1}{4}z^2 + \frac{1}{12}z^3 + \cdots + \frac{1}{12}z^6$

- A, B indep: $P(A\cap B) = P(A)P(B) \Leftrightarrow P(A|B) = P(A)$

X, Y indep: $P(X=a, Y=b) = P(X=a)P(Y=b)$

X	1	2	3	4	5	6
P	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$

Y	5	-2
P	$\frac{5}{12}$	$\frac{7}{12}$

- (全機率) $P(A) = \sum_i P(B_i)P(A|B_i)$

(貝氏定理) $P(B_k|A) = \frac{P(B_k)P(A|B_k)}{\sum_i P(B_i)P(A|B_i)}$

Discrete Random Variables

X	x_0 x_1 x_2 $\cdots$
P	p_0 p_1 p_2 $\cdots$

$X(\Omega)=\{0,1,2,\cdots\}$ $\Longrightarrow$

X	0 1 2 $\cdots$ k $\cdots$
P	p_0 p_1 p_2 $\cdots$ p_k $\cdots$

$P(X=k)=p_k,\quad k=0,1,2,\cdots$

- $$\begin{cases}\mu = E(X) = \sum\limits_{\omega\in\Omega} X(\omega)P(\omega) = \sum\limits_{k\in X(\Omega)} k\,P(X=k) = G'(1) = M'(0)\\ \sigma^2 = Var(X) = E\left((X-\mu)^2\right) = E(X^2)-\mu^2 = G''(1)+G'(1)-\mu^2 = M''(0)-\mu^2\end{cases}$$

- $$\begin{cases}\text{probability g.f.}:\ G_X(z) = E(z^X) = \sum\limits_{\omega\in\Omega} P(\omega)\, z^{X(\omega)} = \sum\limits_{k\in X(\Omega)} p_k z^k \quad (p_k = P(X=k))\\ \text{moment g.f.}\ M_X(t) = G_X(e^t) = \sum\limits_{k\in X(\Omega)} p_k e^{tk} \qquad G_X(1)=M_X(0)=1\end{cases}$$

- $G_X(z) = \sum\limits_{b\in Y(\Omega)} P(Y=b)\, G_{X|b}(z)$

$$\because G_X(z) = \sum_{k\in X(\Omega)} \underbrace{\sum_{b\in Y(\Omega)} P(Y=b)P(X=k|Y=b)}_{p_k\ (\text{全機率})} z^k = \sum_{b\in Y(\Omega)} P(Y=b)\left(\sum_{k\in X(\Omega)} P(X=k|Y=b)\, z^k\right)$$

(8.92)

- **定理 1** $X,Y:\Omega\to\mathbb{R}$

(1) $E(c) = c$　　(2) $Var(c) = 0$

(3) $E(X+Y) = E(X)+E(Y)$　　(4) $Var(X+Y) = Var(X)+Var(Y)+2\,Cov(X,Y)$

(5) $E(cX) = c\,E(X)$　　(6) $Var(cX) = c^2 Var(X)$

$Cov(X,Y) = E\left((X-\mu_X)(Y-\mu_Y)\right) = E(XY)-\mu_X\mu_Y$

證 (3) $E(X+Y) = \sum\limits_{\omega\in X(\Omega)} (X(\omega)+Y(\omega))P(\omega) = \sum\limits_{\omega} X(\omega)P(\omega) + \sum\limits_{\omega} Y(\omega)P(\omega) = E(X)+E(Y)$

(4) $Var(X+Y) = E\left((X+Y)^2\right) - (\mu_X+\mu_Y)^2 = E\left(X^2+2XY+Y^2\right) - \left(\mu_X^2+2\mu_X\mu_Y+\mu_Y^2\right)$

- **定理 2** X,Y indep

(1) $E(XY) = E(X)E(Y) \Rightarrow Cov(X,Y)=0$

(2) $Var(X+Y) = Var(X)+Var(Y)$

(3) $\begin{cases} G_{X+Y}(z) = G_X(z)\,G_Y(z)\\ M_{X+Y}(t) = M_X(t)\,M_Y(t)\end{cases}$

證 $$E(XY) = \sum_{\substack{a\in X(\Omega)\\ b\in Y(\Omega)}} ab\, \overbrace{P(X=a,Y=b)}^{P(X=a)P(Y=b)} = \sum_{a\in X(\Omega)} a\,P(X=a) \sum_{b\in Y(\Omega)} b\,P(Y=b) = E(X)\,E(Y)$$

常見離散隨機分布		μ	σ^2	
	X: x_0 x_1 $\cdots$ x_k $\cdots$ P: p_0 p_1 $\cdots$ p_k $\cdots$			(1) $P(X=x_k)=p_k\ge 0$ (2) $\sum_{k=0}^{\infty} p_k = 1$
Bernoulli 分布 $X\sim Be(p)$	X: 0 1 P: q p $(q=1-p)$	p	pq	$X=\begin{cases}1 & \text{成功}\\ 0 & \text{失敗}\end{cases}$ （只有兩種結果）
二項分布 $X\sim B(n,p)$	$P(X=k)=\binom{n}{k}p^k q^{n-k},\ k=0,1,\ldots,n$	np	npq	(1) $X=X_1+X_2+\cdots+X_n$ (2) i.i.d. $X_i\sim Be(p)$
Poisson 分布 $X\sim P(\lambda)$	$P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!},\ k=0,1,2,\ldots$	λ	λ	(1) n 較大 (≥ 20)，p 較小 (≤ 0.05) (2) $\lambda\approx np$ (3) $\binom{n}{k}p^k q^{n-k}\approx e^{-\lambda}\frac{\lambda^k}{k!}$
幾何分布 $X\sim G(p)$	$P(X=k)=q^{k-1}p,\ k=1,2,\ldots$	$\frac{1}{p}$	$\frac{q}{p^2}$	k = 首次成功次數

- **Example 1** (Bernouilli), Flipping a coin, #H = ?

解

Ω	T	H
X	0	1
P	q	p

$(q=1-p)$

(甲) $\begin{cases} E(X) = 0\cdot q + 1\cdot p = p, \quad E(X^2) = 0^2 q + 1^2 p = p \\ Var(X) = E(X^2) - \mu^2 = p - p^2 = pq \end{cases}$

(乙) $G_X(z) = q + pz$, $\begin{cases} G'(z) = p, & E(X) = G'(1) = p \\ G''(z) = 0, & Var(X) = G''(1) + G'(1) - \mu^2 = p - p^2 \end{cases}$

- **Example 2** (Binomial) Flipping n coins, #H = ?

$p_k = P(X=k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0,1,\dots,n$

解

Ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	3	2	2	1	2	1	1	0
P	p^3	p^2q	p^2q	pq^2	p^2q	pq^2	pq^2	q^3

$\Rightarrow$

X	0	1	2	3
P	q^3	$\binom{3}{1}pq^2$	$\binom{3}{2}p^2q$	p^3

(甲) $G_X(z) = \sum_{k\ge 0} p_k z^k = \sum_{0\le k\le n} \binom{n}{k} p^k q^{n-k} z^k = (pz+q)^n \xLeftarrow[T \to q]{H \to pz} \Omega = (H+T)^n$

$\begin{cases} G'(z) = np(pz+q)^{n-1} \\ G''(z) = n(n-1)p^2(pz+q)^{n-2} \end{cases}$ $\begin{cases} E(X) = G'(1) = np \\ Var(X) = G''(1) + G'(1) - \mu^2 = npq \end{cases}$

(乙) $\begin{cases} X_i : \text{indep} \\ X_i \sim Be(p) \end{cases} \Rightarrow \begin{cases} E(X) = \sum_{i=1}^{n} E(X_i) = np, & G_X(z) = G_{X_1}(z)\cdot G_{X_2}(z)\cdots G_{X_n}(z) = (q+pz)^n \\ Var(X) = \sum_{i=1}^{n} Var(X_i) = npq, \end{cases}$

$X = X_1 + X_2 + \cdots + X_n$

- **Example 3** (Geometric) # tosses to get H ?

解

Ω	$\{H,$	$TH,$	$T^2H,$	$\dots,$	$T^{k-1}H,$	$\dots\}$
X	1	2	3	$\dots$	k	$\dots$
P	p	qp	q^2p	$\dots$	$q^{k-1}p$	$\dots$

$\{H, TH, T^2H, \dots, T^{k-1}H, \dots\} = T^*H = \sum_{k\ge 1} T^{k-1}H$, $\begin{cases} H \to pz \\ T \to qz \end{cases}$

$p_k = P(X=k) = q^{k-1}p, \quad k = 1,2,\dots$

$G_X(z) = pz + qpz^2 + q^2pz^3 + \cdots + q^{k-1}pz^k + \cdots = \dfrac{pz}{1-qz}$ $\quad \begin{cases} \mu = \dfrac{1}{p} \\ \sigma^2 = \dfrac{q}{p^2} \end{cases}$

- **Example 4** # tosses to get HH ?

解 (甲) $\Omega = (T+HT)^*HH = \sum_{k=0}^{\infty} (T+HT)^k HH \xRightarrow[T\to qz]{H\to pz} \sum_{k=0}^{\infty} (qz + pqz^2)^k p^2z^2 = \dfrac{p^2z^2}{1-qz-pqz^2} = S(z)$

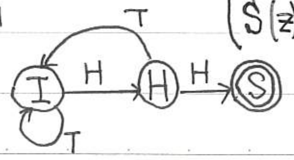
(乙) $\begin{cases} (I) = \{1 + T + HT + \cdots\} \\ \quad = (I)T + (H)T + 1 \\ (H) = (I)H \\ (S) = (H)H \end{cases}$

$\xRightarrow[T\to qz]{H\to pz} \begin{cases} I(z) = I(z)qz + H(z)qz + 1 \\ H(z) = I(z)pz \\ S(z) = H(z)pz \end{cases} \Rightarrow S(z) = \dfrac{p^2z^2}{1-qz-pqz^2}$

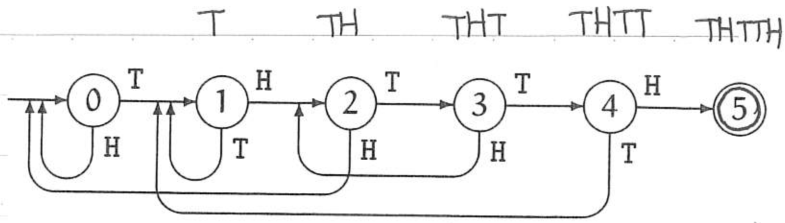
$\begin{cases} \mu = p^{-2} + p^{-1} \\ \sigma^2 = p^{-4} + 2p^{-3} - 2p^{-2} - p^{-1} \end{cases}$

(丙) n tosses

$\begin{cases} I_n = P(\text{at } I) = q I_{n-1} + q H_{n-1} + \delta_{n=0} \\ H_n = P(\text{〃 } H) = p I_{n-1} \\ S_n = P(\text{〃 } S) = p H_{n-1} \end{cases}$ $\begin{cases} I(z) = \sum I_n z^n \\ H(z) = \sum H_n z^n \\ S(z) = \sum_n S_n z^n \end{cases}$



- **Example 5** # tosses for THTTH ?



解 (甲)

$$\begin{cases} S_0 = 1 + S_0 H + S_2 H, \\ S_1 = S_0 T + S_1 T + S_4 T, \\ S_2 = S_1 H + S_3 H, \\ S_3 = S_2 T, \\ S_4 = S_3 T, \\ S_5 = S_4 H. \end{cases} \qquad \begin{cases} N = S_0 + S_1 + S_2 + S_3 + S_4 \\ S = S_5 \end{cases}$$

(乙)

$$\begin{cases} 1 + N(H+T) = N + S, \\ N\,\text{THTTH} = S + S\,\text{TTH}, \end{cases} \Rightarrow \begin{cases} 1 + N(z)(pz+qz) = N(z) + S(z) \\ N(z)\,p^2q^3z^5 = S(z) + S(z)\,pq^2z^3 \end{cases}$$

xxx THTTH

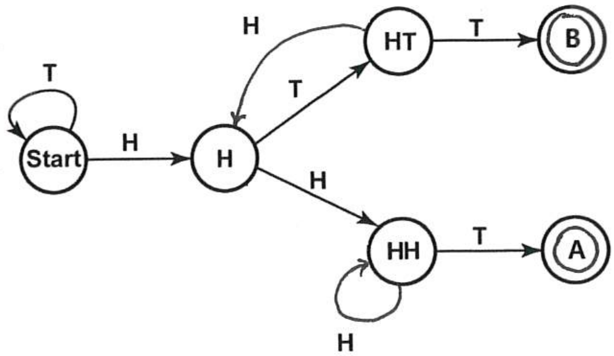
$$\Rightarrow S(z) = \frac{p^2q^3z^5}{p^2q^3z^5 + (1+pq^2z^3)(1-z)}$$

$$\Rightarrow \begin{cases} \mu = p^{-2}q^{-3} + p^{-1}q^{-1} \\ \sigma^2 = \mu^2 - 9p^{-2}q^{-3} - 3p^{-1}q^{-1} \end{cases}$$

General pattern: A = HTHTHHTHTH

$\Rightarrow$ N HTHTHHTHTH = S + S HTHTH + S THHTHTH + S THTHHTHTH

- **Example 6** $\begin{cases} A: \text{HHT} \\ B: \text{HTT} \end{cases}$ 求 $P(A \text{ wins}) = S_A = ?$



解 $(p = q = \frac{1}{2})$

$$\begin{cases} 1 + N(H+T) = N + S_A + S_B \\ N\,\text{HHT} = S_A \\ N\,\text{HTT} = S_A T + S_B \end{cases}$$

$$\xRightarrow[T=\frac{1}{2}]{H=\frac{1}{2}} \begin{cases} 1 + N = N + S_A + S_B \\ N\frac{1}{8} = S_A \\ N\frac{1}{8} = S_A\frac{1}{2} + S_B \end{cases} \Rightarrow \begin{cases} S_A = \frac{2}{3} \\ S_B = \frac{1}{3} \end{cases}$$

General pat. $\begin{cases} A = \text{HTTHTHTH} \\ B = \text{THTHTTH} \end{cases} \Rightarrow \begin{cases} N\,\text{HTTHTHTH} = S_A\,\text{TTHTHTH} + S_A + S_B\,\text{TTHTHTH} + S_B\,\text{THTH} \\ N\,\text{THTHTTH} = S_A\,\text{THTTH} + S_A\,\text{TTH} + S_B\,\text{THTTH} + S_B. \end{cases}$

- **定理** $\tau_2\tau_1\tau_2\cdots\tau_{l-1} > \tau_1\tau_2\cdots\tau_l$.

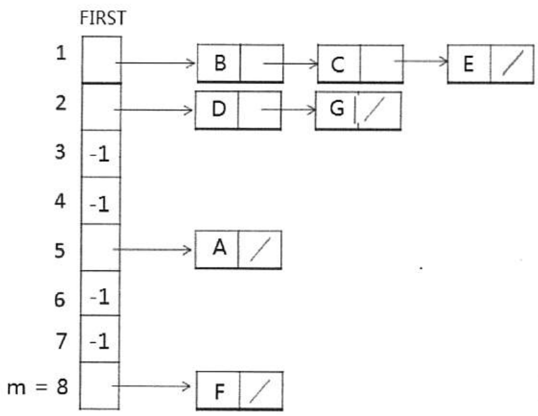
但 HTT > TTH > THH > HHT > HTT ! (De Bruijn sequence)

Analysis of Separate-chaining Hashing Algorithms

(#times executed)

unsucc	**succ**	
		`Search(K) {`
1	1	`    i = hash(K); j = FIRST[i];`
$Y+1$	X	`    while (1) {`
$Y+1$	X	`        if (j <= 0) return(UNSUCC);` ⇒
Y	X	`        if (KEY[j] == K) return(SUCC);`
Y	$X-1$	`        i = j; j = NEXT[j];`
		`    }`
		`}`

```
Insert(K) {
    n = n + 1;
    if (j < 0) FIRST[i] = n;
    else       NEXT[i] = n;
    NEXT[n] = 0; KEY[n] = K;
}
```



	KEY	NEXT
1	D	4
2	A	0
3	B	5
4	G	0
5	C	6
6	E	0
$n \rightarrow$ 7	F	0
8		
9		
10		

`HTable(2,5,1,2,1,1,8)` = hash table from inserting keys D, A, B, G, C, E, F with hash values 2, 5, 1, 2, 1, 1, 8

$$\begin{cases} \frac{1}{7}\sum_{1\le k\le 7} X_7\big((2,5,1,2,1,1,8;\ k)\big) = \frac{1}{7}(1+1+1+2+2+3+1) = \frac{11}{7} \\ \frac{1}{8}\sum_{1\le h\le 8} Y_7\big((2,5,1,2,1,1,8;\ h)\big) = \frac{1}{8}(3+2+0+0+1+0+0+1) = \frac{7}{8} \end{cases}$$

Sample Spaces:

$$\begin{cases} \Omega_{X_n} = \Big\{(h_1,h_2,\ldots,h_n;\ k) \mid\ 1\le h_1,\ldots,h_n\le m,\ \ 1\le k\le n\ \Big\}, & |\Omega_{X_n}| = nm^n \\ \Omega_{Y_n} = \Big\{(h_1,h_2,\ldots,h_n;\ h) \mid\ 1\le h_1,\ldots,h_n\le m,\ \ 1\le h\le m\Big\}, & |\Omega_{Y_n}| = m^{n+1} \end{cases}$$

`HTable`$(h_1,h_2,\ldots,h_n)$ = hash table from inserting keys with hash values $h_1,h_2,\ldots,h_n$

Random Variables: $X_n : \Omega_{X_n} \to \mathbb{R}$; $\quad Y_n : \Omega_{Y_n} \to \mathbb{R}$

$$\begin{cases} X_n\big((h_1,\ldots,h_n;\ k)\big) & = \ \#\text{ probes to } \quad \textbf{succ} \text{ search } k^{\text{th}} \text{ key in } \texttt{HTable}(h_1,\ldots,h_n) \\ Y_n\big((h_1,\ldots,h_n;\ h)\big) & = \ \#\text{ probes to } \textbf{unsucc} \text{ search a key with hash value } h \text{ in } \texttt{HTable}(h_1,\ldots,h_n) \end{cases}$$

$$\begin{cases} \mathbf{E}(X_n) & = \#\text{ probes per } \textbf{succ} \text{ search in a random hash table with } n \text{ keys} \\ & = \displaystyle\sum_{\omega\in\Omega_{X_n}} P(\omega)X_n(\omega) = \frac{1}{nm^n} \sum_{\substack{1\le h_1,\ldots,h_n\le m \\ 1\le k\le n}} X_n(h_1,\ldots,h_n;\ k) \\ \mathbf{E}(Y_n) & = \#\text{ probes per } \textbf{unsucc} \text{ search in a random hash table with } n \text{ keys} \\ & = \displaystyle\sum_{\omega\in\Omega_{Y_n}} P(\omega)Y_n(\omega) = \frac{1}{m^{n+1}} \sum_{\substack{1\le h_1,\ldots,h_n\le m \\ 1\le h\le m}} Y_n(h_1,\ldots,h_n;\ h) \end{cases}$$

Theorem.

$$\begin{cases} \mathbf{E}(X_n) & = 1 + \dfrac{n-1}{2m}; \quad \mathbf{VAR}(X_n) = \dfrac{(n-1)(6m+n-5)}{12m^2} \\ \mathbf{E}(Y_n) & = \dfrac{n}{m}\ ; \quad \mathbf{VAR}(Y_n) = \dfrac{n(m-1)}{m^2} \end{cases}$$

Proof:

(A)

$$\mathbf{E}(Y_n) = \mathbf{E}(\ell_h) = \frac{n}{m}, \quad (\ell_h = \text{length of the } h^{\text{th}} \text{ chain})$$

$$\mathbf{E}(\ell_{\text{hash}(K)}) = 1 + \frac{n-1}{m}$$

$$\mathbf{E}(X_n) = \frac{1+\mathbf{E}(\ell_{\text{hash}(K)})}{2} = 1 + \frac{n-1}{2m} \qquad \blacksquare$$

(B)

$$\begin{aligned} \mathbf{E}(Y_n) &= \frac{1}{m^{n+1}} \sum_{1\le h_1,\ldots,h_n\le m}\ \sum_{1\le h\le m} Y_n(h_1,\ldots,h_n;\ h) \\ &= \frac{1}{m^{n+1}} \sum_{1\le h_1,\ldots,h_n\le m} n \\ &= \frac{1}{m^{n+1}}\, m^n n \\ &= \frac{n}{m} \end{aligned}$$

$$\begin{aligned} \mathbf{E}(X_n) &= \frac{1}{n}\Big[\big(1+\mathbf{E}(Y_0)\big) + \big(1+\mathbf{E}(Y_1)\big) + \cdots + \big(1+\mathbf{E}(Y_{n-1})\big)\Big] \\ &= 1 + \frac{1}{n}\sum_{1\le k\le n} \mathbf{E}(Y_{k-1}) \\ &= 1 + \frac{1}{n}\sum_{1\le k\le n} \frac{k-1}{m} \\ &= 1 + \frac{n-1}{2m} \qquad \blacksquare \end{aligned}$$

(2, 5, 1, 2, 1, 1, 8; 1)

(C) $Y = Z_1 + Z_2 + \cdots + Z_n, \quad Z_j = \delta_{h_j=h} = \begin{cases} 1, & p = 1/m \\ 0, & q = 1-1/m \end{cases} \quad (1 \le j \le n) \quad \text{(i.i.d)}$

$$\Rightarrow G_Y(z) = (q+pz)^n = \Big(\frac{m-1}{m} + \frac{1}{m}z\Big)^n$$

$$\Rightarrow \begin{cases} \mathbf{E}(Y_n) & = & np & = & \dfrac{n}{m} \\ \mathbf{VAR}(Y_n) & = & npq & = & \dfrac{n(m-1)}{m^2} \end{cases}$$

(2, 5, 1, 2, 1, 1, 8; 5)

$X \mid k = Z_1+Z_2+\cdots+Z_k, \quad Z_k = 1, \quad Z_j = \delta_{h_j=h_k} = \begin{cases} 1, & p = 1/m \\ 0, & q = 1-1/m \end{cases} \quad (1 \le j \le k-1)$

$$\Rightarrow \begin{cases} G_{X|k}(z) = \Big(\dfrac{m-1}{m} + \dfrac{1}{m}z\Big)^{k-1} z \\ G_X(z) = \displaystyle\sum_{1\le k\le n} \frac{1}{n} G_{X|k}(z) \qquad\qquad (8.92) \\ \qquad = \displaystyle\sum_{1\le k\le n} \frac{1}{n}\Big(\frac{m-1+z}{m}\Big)^{k-1} z \\ \qquad = \dfrac{z}{n}\,\dfrac{1-\Big(\dfrac{m-1+z}{m}\Big)^n}{1-\dfrac{m-1+z}{m}} \\ \qquad = \dfrac{m}{n}\,\dfrac{z}{1-z}\Big[1-\Big(\dfrac{m-1+z}{m}\Big)^n\Big] \end{cases}$$

$$\Rightarrow \begin{cases} \mathbf{E}(X_n) & = & 1 + \dfrac{n-1}{2m} \\ \mathbf{VAR}(X_n) & = & \dfrac{(n-1)(6m+n-5)}{12m^2} \end{cases} \qquad \blacksquare$$

Lemma. $\mathbf{E}(X_n) = 1 + \dfrac{1}{n}\displaystyle\sum_{1\le k\le n} \mathbf{E}(Y_{k-1})$

Proof:
$$\begin{aligned} \mathbf{E}(X_n) &= \frac{1}{nm^n} \sum_{\substack{1\le h_1,\ldots,h_n\le m \\ 1\le k\le n}} X_n(h_1,\ldots,h_n;\ k) \\ &= \frac{1}{n}\sum_{1\le k\le n} \frac{1}{m^n} \sum_{1\le h_1,\ldots,h_n\le m} X_n(h_1,\ldots,h_n;\ k) \\ &= \frac{1}{n}\sum_{1\le k\le n} \frac{1}{m^n} \sum_{1\le h_1,\ldots,h_k\le m} \Big(1+Y_k(h_1,\ldots,h_{k-1};\ h_k)\Big) m^{n-k} \\ &= \frac{1}{n}\sum_{1\le k\le n} \frac{1}{m^k} \sum_{1\le h_1,\ldots,h_k\le m} \Big(1+Y_k(h_1,\ldots,h_{k-1};\ h_k)\Big) \\ &= 1 + \frac{1}{n}\sum_{1\le k\le n} \mathbf{E}(Y_{k-1}) \qquad \blacksquare \end{aligned}$$

Analysis of Binary Search Trees (BST) Algorithms

Goal: to derive

$$\begin{cases} \mathbb{E}(X_n) & := \text{average \# probes per } \textbf{succ} \text{ search in } n\text{-node BST} \\ \mathbb{E}(Y_n) & := \text{average \# probes per } \textbf{unsucc} \text{ search} \quad \text{"} \end{cases}$$

Sample Spaces:

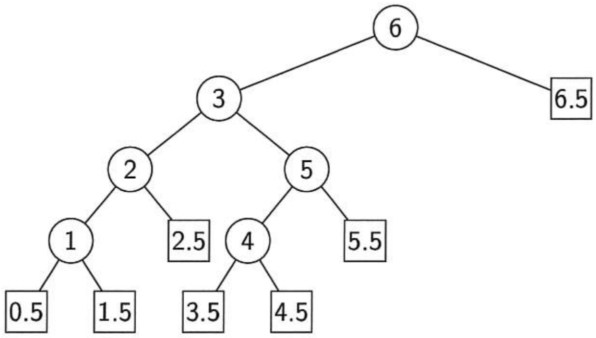
$$\begin{cases} \Omega_{X_n} & = \left\{ (x_1, x_2, \ldots, x_n; k) \,\middle|\, \begin{matrix} (x_1, x_2, \ldots, x_n) \in S_n \\ 1 \le k \le n \end{matrix} \right\}, \quad |\Omega_{X_n}| = n!\, n \\ \Omega_{Y_n} & = \left\{ (x_1, x_2, \ldots, x_n; y) \,\middle|\, \begin{matrix} (x_1, x_2, \ldots, x_n) \in S_n \\ y = 0.5,\, 1.5,\, \ldots,\, n.5 \end{matrix} \right\}, \quad |\Omega_{Y_n}| = (n+1)! \end{cases}$$

$$\begin{aligned} S_n &= \text{set of all permutations (relative order of keys) of } \{1, 2, \ldots, n\} \\ \Omega_{Y_n} &= \{(x_1, x_2, \ldots, x_n; y) \mid (x_1, x_2, \ldots, x_n; y) \in S_{n+1}\}, \quad \text{by re-numbering} \end{aligned}$$

Random Variables:

$$\begin{cases} X_n\big((x_1, x_2, \ldots, x_n; k)\big) & := \text{\# probes to } \textbf{succ} \text{ search } x_k \text{ in } T(x_1, x_2, \ldots, x_n) \\ Y_n\big((x_1, x_2, \ldots, x_n; y)\big) & := \text{\# probes to } \textbf{unsucc} \text{ search } y \text{ in} \quad \text{"} \end{cases}$$

$$\begin{aligned} T(x_1, x_2, \ldots, x_n) &:= \text{BST formed by inserting keys } x_1, x_2, \ldots, x_n \text{ consecutively} \\ T(6, 3, 2, 1, 5, 4) &= \end{aligned}$$



$$\begin{cases} \sum_{1 \le k \le 6} X_6\big((6, 3, 2, 1, 5, 4; k)\big) & = (1+2+3+4+3+4) & = 17 = 6 + I\big(T(6, 3, 2, 1, 5, 4)\big) \\ \sum_{y = 0.5, \ldots, 6.5} Y_6\big((6, 3, 2, 1, 5, 4; y)\big) & = (4+4+3+4+4+3+1) & = 23 = \quad E\big(T(\cdot)\big) \end{cases}$$

$$\begin{cases} I(T) & := \sum_{x:\text{ int node of } T} \text{path length}(x) = 11 \\ E(T) & := \sum_{y:\text{ ext node of } T} \text{path length}(y) = 23 = I(T) + 2 \cdot 6 \end{cases}$$

$$\mathbb{E}(X_6) = \frac{1}{6}\Big[\big(1 + \mathbb{E}(Y_0)\big) + \big(1 + \mathbb{E}(Y_1)\big) + \cdots + \big(1 + \mathbb{E}(Y_4)\big) + \big(1 + \mathbb{E}(Y_5)\big)\Big]$$

n	1	2	3	
$\mathbb{E}(X_n) = 2\frac{n+1}{n}H_n - 3$	1	$\frac{3}{2}$	$\frac{17}{9}$	$\cdots$
$\mathbb{E}(Y_n) = 2H_{n+1} - 2$	1	$\frac{5}{3}$	$\frac{13}{6}$	$\cdots$

$\boxed{n=1}$

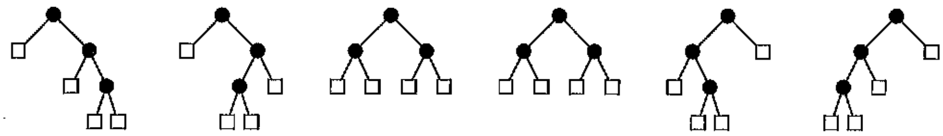
$$\begin{cases} \mathbb{E}(X_1) & = \frac{1}{1}\,[1] & = 1 \\ \mathbb{E}(Y_1) & = \frac{1}{2}\,[1+1] & = 1 \end{cases}$$

$\boxed{n=2}$

(1, 2) (2, 1)

$$\begin{cases} \mathbb{E}(X_2) & = \frac{1}{2\cdot 2!}\,[(1+2)+(1+2)] & = \frac{3}{2} \\ \mathbb{E}(Y_2) & = \frac{1}{3\cdot 2!}\,[(1+2+2)+(2+2+1)] & = \frac{5}{3} \end{cases}$$

$\boxed{n=3}$



(1, 2, 3) (1, 3, 2) (2, 1, 3) (2, 3, 1) (3, 1, 2) (3, 2, 1)

$$\begin{cases} \mathbb{E}(X_3) & = \frac{1}{3\cdot 3!}\,[(1+2+3)+(1+2+3)+(1+2+2)+(\cdot)+(\cdot)+(\cdot)] & = \frac{17}{9} \\ \mathbb{E}(Y_3) & = \frac{1}{4\cdot 3!}\,[(1+2+3+3)+(1+3+3+2)+(2+2+2+2)+(\cdot)+(\cdot)+(\cdot)] & = \frac{13}{6} \end{cases}$$

Lemma 1. An n-node binary tree T has

(1) $2n$ edges,

(2) $n+1$ external nodes (leaves),

(3) $E(T) = I(T) + 2n$.

Lemma 2. $\mathbb{E}(X_n) = 1 + \dfrac{1}{n} \displaystyle\sum_{1 \le k \le n} \mathbb{E}(Y_{k-1})$

Theorem 3.

$$\begin{cases} \mathbb{E}(X_n) & = 2\frac{n+1}{n} H_n - 3 \quad \approx 2 \ln n \\ \mathbb{E}(Y_n) & = 2H_{n+1} - 2 \quad \approx 2 \ln n \end{cases}$$

Proof:

$$\begin{cases} \mathbb{E}(X_n) & = \sum\limits_{w \in \Omega_{X_n}} P(w) X_n(w) = \sum\limits_{\substack{(x_1,\dots,x_n) \in S_n \\ 1 \le k \le n}} \frac{1}{n!\,n} X_n(x_1, \dots, x_n; k) \\ & = \frac{1}{n!\,n} \sum\limits_{(x_1,\dots,x_n) \in S_n} \sum\limits_{1 \le k \le n} X_n(x_1, \dots, x_n; k) \\ & = \frac{1}{n!\,n} \sum\limits_{T} (n + I(T)) \\ & = \frac{1}{n} \left[n + \frac{1}{n!} \sum\limits_{T} I(T) \right] & (1) \\ \mathbb{E}(Y_n) & = \frac{1}{(n+1)!} \sum\limits_{(x_1,\dots,x_n) \in S_n} \sum\limits_{y = 0.5,\dots,n.5} Y_n(x_1, \dots, x_n; y) \\ & = \frac{1}{(n+1)!} \sum\limits_{T} E(T) \\ & = \frac{1}{(n+1)n!} \sum\limits_{T} (2n + I(T)) \\ & = \frac{1}{n+1} \left[2n + \frac{1}{n!} \sum\limits_{T} I(T) \right] & (2) \end{cases}$$

$(1) + (2) \qquad \Rightarrow (n+1)\mathbb{E}(Y_n) = n\mathbb{E}(X_n) + n$

$+$ **Lemma 2** $\Rightarrow$

$$(n+1)\,\mathbb{E}(Y_n) = \sum_{1 \le k \le n} \mathbb{E}(Y_{k-1}) + 2n$$

$$-) \qquad n\,\mathbb{E}(Y_{n-1}) = \sum_{1 \le k \le n-1} \mathbb{E}(Y_{k-1}) + 2(n-1)$$

$$(n+1)\,\mathbb{E}(Y_n) = (n+1)\,\mathbb{E}(Y_{n-1}) + 2$$

$$\Rightarrow \begin{cases} \mathbb{E}(Y_n) & = \mathbb{E}(Y_{n-1}) + \frac{2}{n+1} \\ \mathbb{E}(Y_{n-1}) & = \mathbb{E}(Y_{n-2}) + \frac{2}{n} \\ & \vdots \\ \mathbb{E}(Y_2) & = \mathbb{E}(Y_1) + \frac{2}{3} \\ \mathbb{E}(Y_1) & = \frac{2}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \mathbb{E}(Y_n) & = 2H_{n+1} - 2 \\ \mathbb{E}(X_n) & = \frac{n+1}{n} E(Y_n) - 1 = 2\frac{n+1}{n} H_n - 3 \end{cases} \qquad \blacksquare$$

Lemma 2. $\mathbb{E}(X_n) = 1 + \frac{1}{n} \sum_{1 \le k \le n} \mathbb{E}(Y_{k-1})$

Proof:

$$
\begin{aligned}
\mathbb{E}(X_n) &= \frac{1}{n!\,n} \sum_{(x_1,\dots,x_n)} \sum_{1 \le k \le n} X_n(x_1, \dots, x_n; k) \\
&= \frac{1}{n!\,n} \sum_{(x_1,\dots,x_n) \in S_n} \sum_{1 \le k \le n} \Big(1 + Y_n(x_1, \dots, x_{k-1}; k)\Big) \\
&= 1 + \frac{1}{n!\,n} \sum_{1 \le k \le n} \sum_{(x_1,\dots,x_k) \in S_k} n \cdots (k+1) Y_{k-1}(x_1, \dots, x_{k-1}; x_k) \\
&= 1 + \frac{1}{n} \sum_{1 \le k \le n} \left[\frac{1}{k!} \sum_{(x_1,\dots,x_k) \in S_k} Y_{k-1}(x_1, \dots, x_{k-1}; x_k)\right] \\
&= 1 + \frac{1}{n} \sum_{1 \le k \le n} \mathbb{E}(Y_{k-1}) \qquad \blacksquare
\end{aligned}
$$

Remark:

$$
\begin{cases}
\text{Dynamic model} & : \ n! \text{ trees;} \\
\text{Static} \quad \text{"} & : \ \frac{1}{n+1}\binom{2n}{n} \text{ trees,} \quad (\text{BST } T(6,3,2,1,5,4) \equiv T(6,3,5,4,2,1))
\end{cases}
$$