Chapter 1 Vectors and Matrices

Vector Space $\mathbb{R}^n$

$c, d \in \mathbb{R}$

- $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| x_i \in \mathbb{R} \right\}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $-\mathbf{x} = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix} \in \mathbb{R}^n$

(1) $\mathbb{R}^n$ is vector space over $\mathbb{R}$, 運算 $\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}$, $c\mathbf{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$, (滿足 (1)-(8))

(2) $\mathbb{R}^n$ 〃 inner-product space, 〃 $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_ny_n \in \mathbb{R}$ (滿足 (9)-(12))

基本性質

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$

(2) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$

(3) $\mathbf{0}+\mathbf{x} = \mathbf{x}$

(4) $(-\mathbf{x})+\mathbf{x} = \mathbf{0}$

(5) $1\,\mathbf{x} = \mathbf{x}$

(6) $c(d\mathbf{x}) = (cd)\mathbf{x}$

(7) $c(\mathbf{x}+\mathbf{y}) = c\mathbf{x}+c\mathbf{y}$

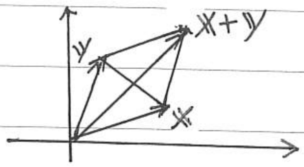
(8) $(c+d)\mathbf{x} = c\mathbf{x}+d\mathbf{x}$

(9) $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$

(10) $(\mathbf{x}+\mathbf{y})\cdot\mathbf{z} = \mathbf{x}\cdot\mathbf{z} + \mathbf{y}\cdot\mathbf{z}$

(11) $(c\mathbf{x})\cdot\mathbf{y} = c(\mathbf{x}\cdot\mathbf{y})$

(12) $\begin{cases} \mathbf{x}\cdot\mathbf{x} \geq 0 \\ \mathbf{x}\cdot\mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{0} \end{cases}$



- 定義 (1) $\mathbf{x}-\mathbf{y} = \mathbf{x}+(-\mathbf{y})$

(2) $\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{x_1^2+\cdots+x_n^2}$

$\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$: unit vector

(3) $\mathbf{x} \parallel \mathbf{y} \Leftrightarrow \mathbf{y} = c\mathbf{x}$

(4) $\mathbf{x} \perp \mathbf{y} \Leftrightarrow \mathbf{x}\cdot\mathbf{y} = 0$

(5) $\theta = \cos^{-1} \dfrac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$ ($\mathbf{x}, \mathbf{y}$ 夾角)

- Property

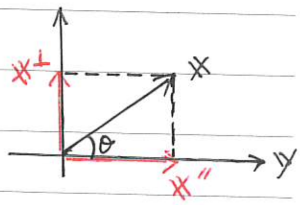
(1) $0\mathbf{x} = c\mathbf{0} = \mathbf{0}$, $(-1)\mathbf{x} = -\mathbf{x}$

(2) (10)+(11) $\Leftrightarrow (c\mathbf{x}+d\mathbf{y})\cdot\mathbf{z} = c(\mathbf{x}\cdot\mathbf{z})+d(\mathbf{y}\cdot\mathbf{z})$ (left linear)

(3) $\mathbf{z}\cdot(c\mathbf{x}+d\mathbf{y}) = c(\mathbf{z}\cdot\mathbf{x})+d(\mathbf{z}\cdot\mathbf{y})$ (right 〃)

(4) $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\,\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2$ $\left(= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})\right)$

$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\,\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2$



(5) $\mathbf{y} \neq \mathbf{0}$, $\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$

$= \dfrac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} + \left(\mathbf{x} - \dfrac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\right)$

其中 $\begin{cases} \mathbf{x}^{\parallel} \parallel \mathbf{y} & \text{(投影向量)} \\ \mathbf{x}^{\perp} \perp \mathbf{y} \end{cases}$

• **定理** (1) (Cauchy 不等式) $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|$

("=" 成立 $\Leftrightarrow$ $\mathbf{x} /\!/ \mathbf{y}$)

$\mathbb{R}^n$: $(x_1y_1+\cdots+x_ny_n)^2 \le (x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)$

$\mathcal{C}^0([a,b])$: $\left|\int_a^b f(t)g(t)\,dt\right|^2 \le \int_a^b f^2(t)\,dt \int_a^b g^2(t)\,dt$

(2) (三角 〃) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\|+\|\mathbf{y}\|$

(3) (算幾 〃) $\dfrac{a^2+b^2}{2} \ge ab$

証 (1) $\left(\|\mathbf{x}\|^2\|\mathbf{y}\|^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2\cos^2\theta = \|\mathbf{x}\|^2\|\mathbf{y}\|^2\sin^2\theta = (\mathbf{x}^\perp\cdot\mathbf{x}^\perp)\|\mathbf{y}\|^2\right)$

$$\left(\mathbf{x}-\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\right)\cdot\left(\mathbf{x}-\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\right) = \|\mathbf{x}\|^2-\frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2} \ge 0$$

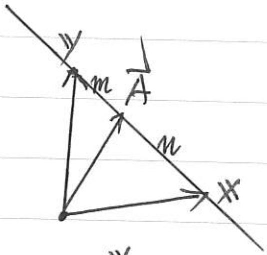
("=" $\Leftrightarrow$ $\mathbf{x} /\!/ \mathbf{y}$ $\Leftrightarrow$ $\mathbf{x}^\perp = \mathbf{0}$)

(2) $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2+2\mathbf{x}\cdot\mathbf{y}+\|\mathbf{y}\|^2 \le \|\mathbf{x}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|+\|\mathbf{y}\|^2$

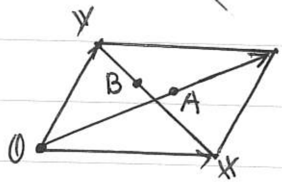
(3) $\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}b\\a\end{bmatrix} \le \sqrt{a^2+b^2}\sqrt{a^2+b^2}$

• **幾何應用**

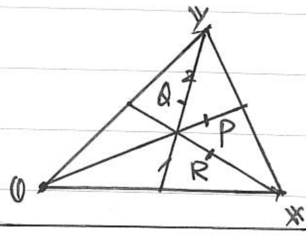
(1) $\vec{A} = \dfrac{m\mathbf{x}+n\mathbf{y}}{m+n}$



(2) 平行▱ 對角線互相平分



(3) △ 三中線交於 $\frac{2}{3}$ 處 (重心)



Pf: (1) $\vec{A} = \mathbf{y}+\dfrac{m}{m+n}(\mathbf{x}-\mathbf{y})$

(2) $\vec{A} = \frac{1}{2}(\mathbf{x}+\mathbf{y})$

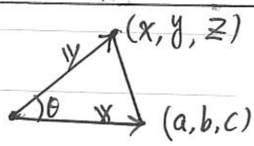
$\vec{B} = \frac{1}{2}(\mathbf{x}+\mathbf{y})$

(3) $\vec{P} = \frac{2}{3}\cdot\frac{1}{2}(\mathbf{x}+\mathbf{y})$

$\vec{Q} = \dfrac{2\cdot\frac{1}{2}\mathbf{x}+\mathbf{y}}{2+1}$

$\vec{R} = \dfrac{\mathbf{x}+2\cdot\frac{1}{2}\mathbf{y}}{2+1}$

• **Remark:**



$\|\mathbf{x}-\mathbf{y}\|^2 \overset{(1)}{=} \|\mathbf{x}\|^2-2\mathbf{x}\cdot\mathbf{y}+\|\mathbf{y}\|^2$

(餘弦) $\overset{(2)}{=} \|\mathbf{x}\|^2-2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta+\|\mathbf{y}\|^2$

(距離) $\overset{(3)}{=} (a-x)^2+(b-y)^2+(c-z)^2$

$= \|\mathbf{x}\|^2-2(ax+by+cz)+\|\mathbf{y}\|^2$

Vector space $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_1, \cdots, x_n \in \mathbb{R} \right\}$

$\mathbb{R}^n$ 中的平直物件 V: $\begin{cases} \text{方程式: } A\mathbb{x} = \mathbb{b} \\ \text{參數式: } \mathbb{x}_0 + \mathrm{Span}(\mathbb{u}_1, \mathbb{u}_2, \cdots, \mathbb{u}_k) \end{cases}$

(1) $\boxed{\text{方程式}}$: Hyperplane in $\mathbb{R}^n$ (dim = $n-1$)

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 & : p_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 & : p_2 \\ \vdots & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m & : p_m \end{cases} \Leftrightarrow \begin{cases} \mathbb{a}_1 \cdot \mathbb{x} = b_1 \\ \mathbb{a}_2 \cdot \mathbb{x} = b_2 \\ \vdots \\ \mathbb{a}_m \cdot \mathbb{x} = b_m \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbb{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbb{b}}$$

- 問題 $\begin{cases} (1) \; \mathbb{x} \in V\,? \\ (2) \; \text{參數式} = ? \end{cases}$ (解線性聯立方程式: 高斯消去法)

- **Example 1** $\mathbb{R}^5$: $x_1 + x_2 - x_3 + 2x_4 + x_5 = 2$ Hyperplane, (dim = 4)

$$\mathbb{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - x_2 + x_3 - 2x_4 - x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- **Example 2** $\mathbb{R}^3$: $\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ x_1 - x_2 + 2x_3 = 5 \end{cases}$ (dim = 1)

$$\mathbb{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - x_3 \\ -1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \mathbb{x}_0 + t\,\mathbb{u}$$

- **Example 3** $\mathbb{R}^3$: $x_1 - 2x_2 + 5x_3 = 3$ (dim = 2)

$$\mathbb{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 + 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \mathbb{x}_0 + s\,\mathbb{u} + t\,\mathbb{v}$$

- 定理: Plane in $\mathbb{R}^3$

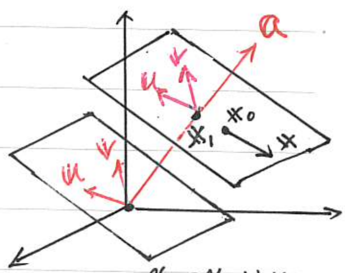
(1) $\mathbb{a} \cdot (\mathbb{x} - \mathbb{x}_0) = 0$ $\mathbb{a}$: 法向量

$\mathbb{a} \cdot \mathbb{x} = \mathbb{a} \cdot \mathbb{x}_0 = d \Leftrightarrow ax + by + cz = d$

(2) $\dfrac{\mathbb{a}}{\|\mathbb{a}\|} \cdot \mathbb{x} = \dfrac{d}{\|\mathbb{a}\|}$ ($\mathbb{0}$ 到平面距離)

(3) $\mathbb{x}_1 = \dfrac{d}{\|\mathbb{a}\|} \dfrac{\mathbb{a}}{\|\mathbb{a}\|}$ (垂足)

(4) $\begin{cases} \mathbb{a} \perp \mathbb{u} \\ \mathbb{a} \perp \mathbb{v} \end{cases}$



$x_1 - 2x_2 + 5x_3 = 0$

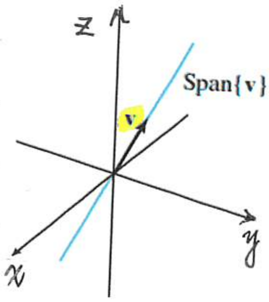
(2) $\boxed{\text{參數式}}$

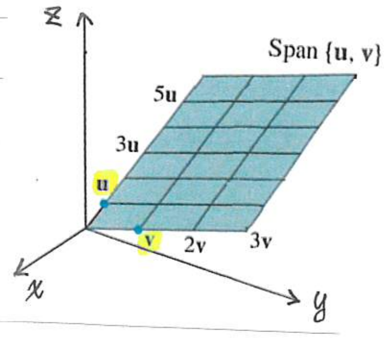
linear span　　linear combination (線性組合) of $\begin{matrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_k \end{matrix}$

$V: \mathbf{x}_0 + \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$　　$\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \{c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$

$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k$　　Given $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$

- $\mathrm{Span}(\mathbf{v}) = \{c\mathbf{v} \mid c \in \mathbb{R}\}$　　$\dim = 1.$
- $\mathrm{Span}(\mathbf{u}, \mathbf{v}) = \{c\mathbf{u} + d\mathbf{v} \mid c, d \in \mathbb{R}\}$　　$\dim = \begin{cases} 1 & \mathbf{u} \parallel \mathbf{v} \\ 2 & \text{else} \end{cases}$





- 問題 $\begin{cases} (1)\ \mathbf{b} \in V \ ? \ \Leftrightarrow \ \exists\, c_1, c_2, \dots, c_k \in \mathbb{R} \ \ \mathbf{b} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k \\ (2)\ V: \text{方程式} = ? \quad \begin{cases} \text{(i) (ii) 求 } \mathbf{a} \perp \mathbf{v}_i, \ 1 \le i \le k \\ \text{(iii) constraint equation} \end{cases} \end{cases}$

- Example 1: $\mathbb{R}^4$

(甲) $\begin{bmatrix} 5 \\ 6 \\ 3 \\ 1 \end{bmatrix} \in \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right) = \mathbf{x}_0 + \mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$?

$\Leftrightarrow \begin{bmatrix} 5 \\ 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ 有解 c_1, c_2, c_3

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = 0 \\ c_3 = 3 \end{cases} \quad \because \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(乙) $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbf{x}_0 + \mathrm{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a-1 \\ b-3 \\ c-2 \\ d+1 \end{bmatrix} \quad \because \left[\begin{array}{ccc|c} 1 & 1 & 2 & a-1 \\ 0 & 1 & 1 & b-3 \\ 1 & 1 & 1 & c-2 \\ 2 & 1 & 2 & d+1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & a-1 \\ 0 & 1 & 1 & b-3 \\ 0 & 0 & 1 & a-c+1 \\ 0 & 0 & 0 & \underline{-a+b-c+d+1} \end{array}\right]$$

$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \perp \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, $\begin{cases} \text{有解} \Leftrightarrow -a+b-c+d+1 = 0 \\ \text{方程式}: \ \underline{-x_1 + x_2 - x_3 + x_4 + 1 = 0} \end{cases}$

- Example 2: $\mathbb{R}^4$

$\mathrm{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 3 \end{bmatrix} \right)$ 方程式？

解: $\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 3 & 2 & -1 & b \\ 1 & 4 & -3 & c \\ 3 & -3 & 3 & d \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 5 & -4 & -3a+b \\ 0 & 0 & 0 & \underline{2a-b+c} \\ 0 & 0 & 0 & \underline{-3a+d} \end{array}\right]$

- 有解 $\Leftrightarrow \begin{cases} 2a - b + c = 0 \\ -3a + d = 0 \end{cases}$ 或 $\begin{cases} 2x_1 - x_2 + x_3 = 0 \\ -3x_1 + x_4 = 0 \end{cases}$

- $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \perp \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 3 \end{bmatrix} \right\}$

線性聯立方程式　(1) Gauß Elimination + (2) Jordan Reduction

- 高斯消去法　Augmented matrix $\rightsquigarrow$ Echelon matrix
  3 elementary row ops
- 線性聯立方程式

$$\begin{cases} x_1 + x_2 + 3x_3 - x_4 = 0 \\ -x_1 + x_2 + x_3 + x_4 + 2x_5 = -4 \\ x_2 + 2x_3 + 2x_4 - x_5 = 0 \\ 2x_1 - x_2 + x_4 - 6x_5 = 9 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 2 & -1 & 0 & 1 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ -4 \\ 0 \\ 9 \end{bmatrix}}_{b}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 2 & -4 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 & -6 & 9 \end{array}\right] \begin{array}{l} \\ \times 1 \\ \\ \times(-2) \end{array}$$

$$\xrightarrow[(-2)R_1+R_4]{1R_1+R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 2 & -4 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -3 & -6 & 3 & -6 & 9 \end{array}\right] \begin{array}{l} \\ \times \frac{1}{2} \\ \\ \times(-\frac{1}{3}) \end{array}$$

$$\xrightarrow[(-\frac{1}{3})R_4]{\frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ \times(-1) \\ \times(-1) \end{array}$$

$$\xrightarrow[(-1)R_2+R_4]{(-1)R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right] \begin{array}{l} \\ \\ \times(-\frac{1}{2}) \\ \times 1 \end{array}$$

free variables

$$\xrightarrow[1R_3+R_4]{(-\frac{1}{2})R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Echelon form} \quad \begin{cases} x_1 + x_2 + 3x_3 - x_4 = 0 \\ x_2 + 2x_3 + x_5 = -2 \\ x_4 - x_5 = 1 \end{cases}$$

Jordan Reduction

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -2 & 3 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Reduced Echelon form} \quad \begin{cases} x_1 + x_3 - 2x_5 = 3 \\ x_2 + 2x_3 + x_5 = -2 \\ x_4 - x_5 = 1 \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 - x_3 + 2x_5 \\ -2 - 2x_3 - x_5 \\ x_3 \\ 1 + x_5 \\ x_5 \end{bmatrix}}_{(\text{一般解})} = \underbrace{\begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{(\text{特別解})} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$\begin{cases} \text{沒自由變量} \Rightarrow \text{唯一解} \\ \text{有 〃} \Rightarrow \infty \text{ 解} \end{cases}$

- Gauß Elimination:
  - Elementary row operations: $\begin{cases} R_{ij} \\ cR_i \\ cR_i + R_j \end{cases}$ (沒改變解) 消去多餘列 $\begin{cases} R_i \leftrightarrow R_j \\ R_i \leftarrow cR_i \\ R_j \leftarrow cR_i + R_j \end{cases}$
  - Echelon matrix (form)
    (1) pivot (leading term) moves right in succession
    (2) all "0" below pivots
    (3) all "0" rows at bottom
  - Reduced Echelon matrix
    (1) leading terms = 1
    (2) "0" above pivots

- Example 2

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -3 & 7 \\ 2 & 1 & 1 & 0 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 3 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & \underset{0}{\cancel{5}} \end{array}\right] \equiv \begin{cases} x_1 - x_2 + 2x_3 + 3x_4 = 2 \\ 3x_2 - 3x_3 - 6x_4 = -3 \\ 0 = 8 \end{cases}$$

(Inconsistent)

Remarks (1) 線性方程組的解只有化成 Echelon form 才知道

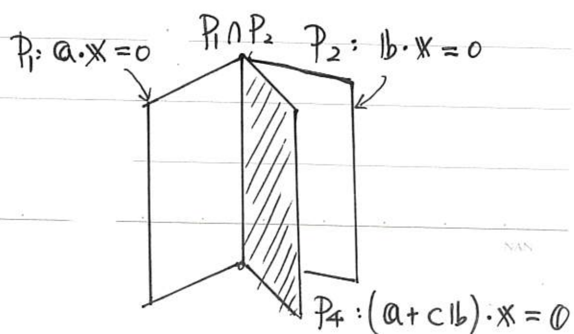
(2) 〃 理論, 線代核心課題之一

(3) $\begin{cases} P_1: & x + y + z = 3 \\ P_2: & 4x + 3y + 2z = 9 \\ P_3: & x + y - z = 1 \end{cases} \Rightarrow P_4 = (-1)P_1 + P_2 : 3x + 2y + z = 6$

$\begin{cases} \text{depends on } P_1, P_2 \text{ (非獨立於} P_1, P_2\text{)} \\ \text{經過 } P_1 \cap P_2 \\ \text{no extra information} \end{cases}$

$\rightsquigarrow \begin{cases} P_5: & x & & = 1 \\ P_6: & & y & = 1 \\ P_7: & & z & = 1 \end{cases}$

$\begin{cases} \text{rank} = 3 \\ 3 \text{ indep 方程式} \end{cases}$



$$\begin{cases} \mathbb{b} = \mathbb{0} & \text{Homogeneous} \\ \mathbb{b} \neq \mathbb{0} & \text{Inhomogeneous} \end{cases}$$

Theory of Linear equations

$$A \quad \mathbb{x} = \mathbb{b}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \iff \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$[A|\mathbb{b}]$: augmented matrix

- **Gauß消去法**

Pivot　　　　自由變數

$$[A|\mathbb{b}] \rightsquigarrow \underbrace{\left[\begin{array}{cccccccccc|c} c_1 & \times & 0 & \times & 0 & 0 & \times & \times & 0 & \times & d_1 \\ 0 & 0 & c_2 & \times & 0 & 0 & \times & \times & 0 & \times & d_2 \\ 0 & 0 & 0 & 0 & c_3 & 0 & \times & \times & 0 & \times & d_3 \\ 0 & 0 & 0 & 0 & 0 & c_4 & \times & \times & 0 & \times & d_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_r & \times & d_r \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{r+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{m\times n} m\times(n+1) = [C|\mathbb{d}] \begin{cases} \text{Echelon matrix} \\ \Downarrow \\ \text{Reduced} \quad \text{"} \end{cases}$$

$$\begin{cases} \operatorname{rank}(A) = 5 \\ \operatorname{rank}([A|\mathbb{b}]) = 6 \end{cases}$$

- **定義**: $\operatorname{rank}(A) := r = \#\text{pivots} = \#\text{non-zero rows} = \#\text{indep rows}$ (equations)

$$A_{n\times n}: \begin{cases} \text{nonsingular} \iff \operatorname{rank}(A) = n \quad \text{(regular)} \\ \text{singular} \iff \qquad < n \end{cases}$$

- **定理** $A\mathbb{x} = \mathbb{b}$ (0-1-∞ rule)

(1) $d_{r+1} \neq 0 \Rightarrow$ 無解 (inconsistent), $\operatorname{rank}[A|\mathbb{b}] = \operatorname{rank}[A] + 1$

(2) $d_{r+1} = 0,\ r = n \Rightarrow$ 唯一解　" $= \operatorname{rank}[A] = n$

(3) $d_{r+1} = 0,\ r < n \Rightarrow \infty$ 解　" $=$ " $< n$
(n-r個自由變數)

$A\mathbb{x} = \mathbb{0}$:
(不存在)
唯一解 $\mathbb{x} = \mathbb{0}$
∞ 解

- **Proposition**

(1) $\operatorname{rank}[A|\mathbb{b}] = \operatorname{rank}[A]$ $(d_{r+1} = 0) \iff A\mathbb{x} = \mathbb{b}$ consistent (有解)

(2) $\operatorname{rank}[A] = m \Rightarrow A\mathbb{x} = \mathbb{b}$ consistent

(3) $A\mathbb{x} = \mathbb{0}$. consistent

(4) $\begin{cases} A\mathbb{x} = \mathbb{b} \text{ 唯一解} \iff A\mathbb{x} = \mathbb{0} \text{ 唯一解 } \mathbb{x} = \mathbb{0} \\ A\mathbb{x} = \mathbb{b} \ \infty \text{ 解} \iff A\mathbb{x} = \mathbb{0} \ \infty \text{ 解} \end{cases}$

- **Proposition** $\begin{cases} P = \{ \mathbf{x} \mid A\mathbf{x} = \mathbf{b} \} \\ Q = \{ \mathbf{y} \mid A\mathbf{y} = \mathbf{0} \} \\ \mathbf{x}_0 \in P \end{cases} \Rightarrow P = \mathbf{x}_0 + Q = \{ \mathbf{x}_0 + \mathbf{y} \mid \mathbf{y} \in Q \}$

$$\underbrace{\mathbf{x}}_{\text{一般解}} = \underbrace{\mathbf{x}_0}_{\text{特別解}} + \underbrace{\mathbf{y}}_{\text{一般解(通解)}}$$

$$\underbrace{\text{一般解} \quad \text{特別解}}_{A\mathbf{x}=\mathbf{b}} \qquad \underbrace{\text{一般解(通解)}}_{A\mathbf{y}=\mathbf{0}}$$

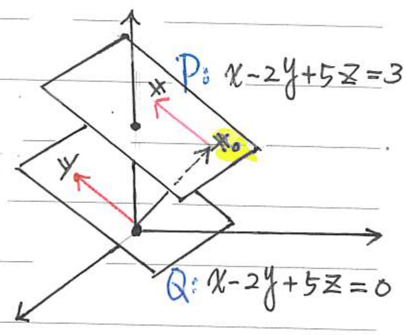
證

"$\supset$": $A(\mathbf{x}_0 + \mathbf{y}) = A\mathbf{x}_0 + A\mathbf{y} = \mathbf{b}$

"$\subset$": $\begin{cases} A\mathbf{x} = \mathbf{b} \Rightarrow A(\mathbf{x} - \mathbf{x}_0) = \mathbf{0} \Rightarrow \mathbf{x} - \mathbf{x}_0 = \mathbf{y} \in Q \\ \quad (\because A\mathbf{x}_0 = \mathbf{b}) \end{cases}$

例 $P: \begin{cases} x - 2y + 5z = 3 \\ \mathbf{x}_0 + \text{span}(\mathbf{u}, \mathbf{v}) \end{cases}$, $\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 3 + 2y - 5z \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_0} + y \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}} + z \underbrace{\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$

$Q: \begin{cases} x - 2y + 5z = 0 \\ \text{span}(\mathbf{u}, \mathbf{v}) \end{cases}$ $\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{y}} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$



- **定理** (rank $= n$) (c)

$A_{n \times n}$: nonsingular $\Leftrightarrow$ (1) $A \rightsquigarrow I_n$ (b)

(2) $A\mathbf{x} = \mathbf{0}$ 只有 $\mathbf{0}$ 解 (d)

(3) $A\mathbf{x} = \mathbf{b}$ 有唯一解 $(A^{-1}\mathbf{b})$ (g)

(4) A^{-1} 存在 (e)

(5) $\det(A) \neq 0$ (Cramer 定理)(Chap.5) (t)

Nonsingular

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\text{Col } A = \mathbb{R}^n$

o. $\dim \text{Col } A = n$

p. $\text{rank } A = n$

q. $\text{Nul } A = \{\mathbf{0}\}$

r. $\dim \text{Nul } A = 0$

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

u. $(\text{Col } A)^{\perp} = \{\mathbf{0}\}$.

v. $(\text{Nul } A)^{\perp} = \mathbb{R}^n$.

w. $\text{Row } A = \mathbb{R}^n$.

x. A has n nonzero singular values.