Chapter 2 Matrix Algebra

Matrix

$\mathcal{M}_{m\times n} = \{A_{m\times n}\}$ Vector space of $m\times n$ matrices $\overset{(1)+(2)}{\Rightarrow}$

$$A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & & a_n \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m\times n} \end{array} \begin{array}{l} \leftarrow A_1 \\ \leftarrow A_2 \\ \\ \leftarrow A_m \end{array}$$

- square $(m=n)$
- diagonal
- $\begin{cases} \text{upper triangular} \\ \text{lower} \end{cases}$
- zero matrix $O_{m\times n}$, O_n
- Identity I_n

• Operations

$A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $c \in \mathbb{R}$

- $A = B$: $a_{ij} = b_{ij},\ \forall i,j$
- $A+B = [a_{ij}+b_{ij}]_{m\times n}$
- $cA = [c\,a_{ij}]_{m\times n}$
- $A_{m\times n}B_{n\times p} = [u_{ij}]_{m\times p}$: $u_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj} = A_i^T \cdot b_j$
- $A^T = [u_{ij}]_{n\times m}$: $u_{ij} = a_{ji}$ (transpose)

Property

(1) $A+B = B+A$

$(A+B)+C = A+(B+C)$

$A+O = A$

$A+(-A) = O$

(2) $c(dA) = (cd)A$

$c(A+B) = cA+cB$

$(c+d)A = cA+dA$

$1A = A$

(3) $AI_n = A = I_mA$

$(A+A')B = AB+A'B$

$A(B+B') = AB+AB'$

$(cA)B = c(AB) = A(cB)$

$(AB)C = A(BC)$ (*)

(4) $AB \neq BA$

- BA undefined: $A_{3\times2}\ B_{2\times1}$
- $A_{3\times2}B_{2\times3} \neq B_{2\times3}A_{3\times2}$
- $A_{2\times2}B_{2\times2} \neq B_{2\times2}A_{2\times2}$

$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix} \neq \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} = BA$

矩陣乘法：

(1) 內積　　(2) 線性組合

$$(1)\quad \begin{array}{c}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\\-1&-2&-3\end{bmatrix}\\ \quad a_1\quad a_2\quad a_3\end{array}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1x+2y+3z\\4x+5y+6z\\7x+8y+9z\\-x-2y-3z\end{bmatrix}=x\begin{bmatrix}1\\4\\7\\-1\end{bmatrix}+y\begin{bmatrix}2\\5\\8\\-2\end{bmatrix}+z\begin{bmatrix}3\\6\\9\\-3\end{bmatrix}$$

$$A\ \mathbb{b}\qquad\qquad = x\,a_1+y\,a_2+z\,a_3$$

$$(2)\quad \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\\-1&-2&-3\end{bmatrix}\begin{bmatrix}x&a&m\\y&b&n\\z&c&p\end{bmatrix}=\begin{bmatrix}1x+2y+3z&1a+2b+3c&1m+2n+3p\\4x+5y+6z&4a+5b+6c&4m+5n+6p\\7x+8y+9z&7a+8b+9c&7m+8n+9p\\-x-2y-3z&-a-2b-3c&-m-2n-3p\end{bmatrix}$$

$$AB=A\begin{bmatrix}\mathbb{b}_1&\mathbb{b}_2&\mathbb{b}_3\end{bmatrix}=\begin{bmatrix}A\mathbb{b}_1&A\mathbb{b}_2&A\mathbb{b}_3\end{bmatrix}$$

(3) ◦ 行

$$A\mathbb{b}=\begin{bmatrix}a_1&a_2&\cdots&a_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=x_1a_1+x_2a_2+\cdots+x_na_n$$

$$AB=A\begin{bmatrix}\mathbb{b}_1&\mathbb{b}_2&\cdots&\mathbb{b}_p\end{bmatrix}=\begin{bmatrix}A\mathbb{b}_1&A\mathbb{b}_2&\cdots&A\mathbb{b}_p\end{bmatrix}$$

◦ 列

$$\begin{bmatrix}x_1&x_2&\cdots&x_n\end{bmatrix}\begin{bmatrix}—B_1—\\—B_2—\\\vdots\\—B_n—\end{bmatrix}=x_1B_1+x_2B_2+\cdots+x_nB_n$$

$$AB=\begin{bmatrix}A_1\\A_2\\\vdots\\A_m\end{bmatrix}B=\begin{bmatrix}A_1B\\A_2B\\\vdots\\A_mB\end{bmatrix}$$

• (※) $(AB)C=A(BC)$

$$=A\begin{bmatrix}\mathbb{b}_1&\mathbb{b}_2&\cdots&\mathbb{b}_p\end{bmatrix}\begin{bmatrix}0&0&\begin{matrix}c_1\\c_2\\\vdots\\c_p\end{matrix}&0\end{bmatrix}$$

證

$$左=\begin{bmatrix}A\mathbb{b}_1&A\mathbb{b}_2&\cdots&A\mathbb{b}_p\end{bmatrix}\begin{bmatrix}0&0&\begin{matrix}c_1\\c_2\\\vdots\\c_p\end{matrix}&0\end{bmatrix}=\begin{bmatrix}0&0&c_1A\mathbb{b}_1+c_2A\mathbb{b}_2+\cdots+c_pA\mathbb{b}_p&0\end{bmatrix}$$

$$右=A\begin{bmatrix}0&0&c_1\mathbb{b}_1+c_2\mathbb{b}_2+\cdots+c_p\mathbb{b}_p&0\end{bmatrix}=\begin{bmatrix}0&0&A(c_1\mathbb{b}_1+c_2\mathbb{b}_2+\cdots+c_p\mathbb{b}_p)&0\end{bmatrix}$$

Block Partition of Matrix

$$A=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{pmatrix},\quad B=\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{st} \end{pmatrix}$$

A 的行分法
= B 的列 〃

$$\Rightarrow \quad AB=\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ \vdots & \vdots & & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rt} \end{pmatrix},$$

$$C_{ij}=\sum_{k=1}^{s} A_{ik}B_{kj} \quad \forall i,j$$

Example 1

$$\underset{\substack{4\times 6\\ [2\times 3]}}{\left[\begin{array}{cc|ccc|c} 1 & 2 & 0 & 2 & -1 & 1 \\ 3 & 1 & -1 & 3 & 2 & 0 \\ \hline 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 1 \end{array}\right]}
\underset{\substack{6\times 3\\ [3\times 2]}}{\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & -3 & 1 \\ \hline -1 & 1 & -2 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \\ \hline 2 & 1 & 1 \end{array}\right]}
=\underset{\substack{4\times 3\\ [2\times 2]}}{\left[\begin{array}{cc|c} 1+4+0+6-1+2 & 0-6+0+4+0+1 & 2 \\ 3+2+1+9+2+0 & 0-3-1+6+0+0 & 20 \\ \hline -2 & -1 & 5 \\ 5 & 0 & 10 \end{array}\right]}$$

$$=\left[\begin{array}{c|c} \begin{bmatrix}1&2\\3&1\end{bmatrix}\begin{bmatrix}1&0\\2&-3\end{bmatrix}+\begin{bmatrix}0&2&-1\\-1&3&2\end{bmatrix}\begin{bmatrix}-1&1\\3&2\\1&0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}2&1\end{bmatrix} & \begin{bmatrix}1&2\\3&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}0&2&-1\\-1&3&2\end{bmatrix}\begin{bmatrix}-2\\0\\4\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\end{bmatrix} \\ \hline \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\2&-3\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&0&3\end{bmatrix}\begin{bmatrix}-1&1\\3&2\\1&0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}2&1\end{bmatrix} & \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&0&3\end{bmatrix}\begin{bmatrix}-2\\0\\4\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\end{bmatrix} \end{array}\right]$$

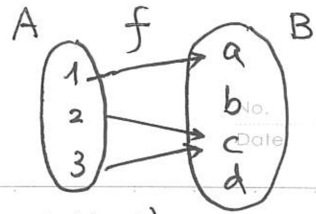
$$=\left[\begin{array}{c|c} \begin{bmatrix}1+4&0-6\\3+2&0-3\end{bmatrix}+\begin{bmatrix}0+6-1&0+4+0\\1+9+2&-1+6+0\end{bmatrix}+\begin{bmatrix}2&1\\0&0\end{bmatrix} & \begin{bmatrix}5\\10\end{bmatrix}+\begin{bmatrix}-4\\10\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix} \\ \hline \begin{bmatrix}1&0\\2&-3\end{bmatrix}+\begin{bmatrix}-3&-1\\1&2\end{bmatrix}+\begin{bmatrix}0&0\\2&1\end{bmatrix} & \begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}2\\8\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix} \end{array}\right]$$

$$=\underset{\substack{4\times 3\\ [2\times 2]}}{\left[\begin{array}{c|c} \begin{bmatrix}1+4+0+6-1+2&0-6+0+4+0+1\\3+2+1+9+2+0&0-3-1+6+0+0\end{bmatrix} & \begin{bmatrix}2\\20\end{bmatrix} \\ \hline \begin{bmatrix}-2&-1\\5&0\end{bmatrix} & \begin{bmatrix}5\\10\end{bmatrix} \end{array}\right]}$$

Example 2 (Projection matrix)

$$P_V = q_1q_1^T+\cdots+q_kq_k^T=\begin{bmatrix} | & & | \\ q_1 & \cdots & q_k \\ | & & | \end{bmatrix}\begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_k^T & - \end{bmatrix}=QQ^T$$

(orthonormal)



$\boxed{\text{函數}}$: $\forall x \in A$, $\exists 1\ y \in B$ 與 x 对應,

A : 定義域

B : 对應域

$f(x) = y$, x 的函數值

$Im(f) = f(A) = \{ f(a) \mid a \in A \}$ 值域

(1) $f : A \longrightarrow B$
$\quad\ \ x \longmapsto y$

(2) f : 1-1 (1 对 1) : $[a \neq b,\ f(a) \neq f(b)] \Leftrightarrow [f(a) = f(b) \Rightarrow a = b]$ (injective)

onto (映成) : $f(A) = B \Leftrightarrow \forall y \in B,\ \exists x,\ f(x) = y$ (surjective)

1-1 & onto (bijective)

(3) 合成函數

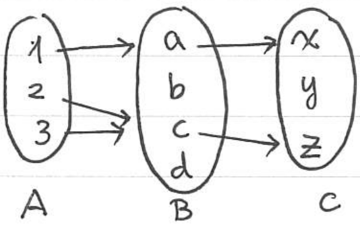
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$(g \circ f) : A \longrightarrow C$

$(g \circ f)(1) = g(f(1)) = g(a) = x$

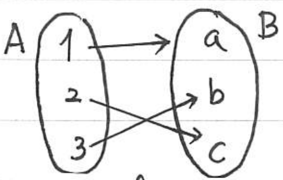
$(g \circ f)(2) = z$

$(g \circ f)(3) = z$



(4) 反函數

f : 1-1, onto



$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$

$f : A \longrightarrow B$
$1 \longmapsto a$
$2 \longmapsto c$
$3 \longmapsto b$

反函數 $f^{-1} : B \longrightarrow A$
$a \longmapsto 1$
$b \longmapsto 3$
$c \longmapsto 2$

$$\begin{cases} f^{-1}(f(x)) = f^{-1}(y) = x, & f^{-1} \circ f = I_A \\ f(f^{-1}(y)) = f(x) = y, & f \circ f^{-1} = I_B \end{cases}$$

(5) $\begin{cases} f \\ g \end{cases} : A \longrightarrow B \subset \mathbb{R}^m$.

定義 $\begin{cases} (f+g) \\ (cf) \end{cases} : A \longrightarrow B$ $\quad \begin{matrix} x \longmapsto f(x) + g(x) \\ x \longmapsto c f(x) \end{matrix}$

$$\begin{bmatrix} (f+g)(x) = f(x) + g(x) \\ (cf)(x) = c f(x) \end{bmatrix}$$

$\mathbb{R}^n$ 基底 $\mathcal{E}=\left\{\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\cdots,\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}\right\}$
$\quad\ \ e_1 \quad e_2 \quad \cdots \quad e_n$

Linear transformation (線性變換)

• **定義**:

$$\begin{cases}(1)\ T:\mathbb{R}^n\to\mathbb{R}^m \text{ is linear (transformation)} \overset{def}{\Longleftrightarrow} \begin{cases}(i)\ T(x+y)=T(x)+T(y)\\(ii)\ T(ax)=aT(x)\end{cases}\\(2)\ \mathcal{L}(\mathbb{R}^n,\mathbb{R}^m):=\{T:\mathbb{R}^n\to\mathbb{R}^m \mid T:\text{linear}\}\end{cases}$$

$$\forall\begin{cases}x,y\in\mathbb{R}^n\\a,b\in\mathbb{R}\end{cases}$$

• **性質**

(1) $(i)+(ii)\Leftrightarrow T(ax+by)=aT(x)+bT(y)$

(2) $T(0_n)=0_m$

(3) $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ is vector space

證 (1) "$\Rightarrow$" $T(ax+by)=T(ax)+T(by)=aT(x)+bT(y)$

(2) $T(x)=T(x+0)=T(x)+T(0)$ (或 $a=0$)

(3) $T_1,T_2\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)\Rightarrow(T_1+T_2)\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$

$$\because\begin{cases}(T_1+T_2)(x+y)=T_1(x+y)+T_2(x+y)=(T_1(x)+T_1(y))+(T_2(x)+T_2(y))\\(T_1+T_2)(x)+(T_1+T_2)(y)\qquad\qquad=(T_1(x)+T_2(x))+(T_1(y)+T_2(y))\end{cases}$$

• **定理**

線性變換(定義) $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ $\qquad$ 矩陣(計算) $\mathcal{M}_{m\times n}$

$$T \xleftarrow{T(x):=Ax} A$$

$(T_A,\ \mu_A)$ $\quad A:=[a_1\cdots a_n]$, $\begin{cases}a_1=T(e_1)\\\ \ \vdots\\a_n=T(e_n)\end{cases}$

使得 $T(x)=Ax$

$$\begin{array}{ccc}\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m) & & \mathcal{M}_{m\times n}\\ aT_A+bT_B & \leftrightarrow & aA+bB\\ T_A\circ T_B & \leftrightarrow & AB\\ T_A^{-1} & \leftrightarrow & A^{-1}\\ (m=n) & & \end{array}$$

證
$$\begin{cases}T\left(\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}\right)=T(x_1e_1+\cdots+x_ne_n)=x_1T(e_1)+\cdots+x_nT(e_n)\\A\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=[a_1\cdots a_n]\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=x_1a_1+\cdots+x_na_n\end{cases}$$

• **Example 1** $T:\mathbb{R}^2\to\mathbb{R}^2$, $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x+y\\x+3y\end{bmatrix}$ 是線性, $A=?$

證1)
$$\begin{cases}T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\right)=\begin{bmatrix}2(x_1+x_2)+(y_1+y_2)\\(x_1+x_2)+3(y_1+y_2)\end{bmatrix}\\T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=\begin{bmatrix}2x_1+y_1\\x_1+3y_1\end{bmatrix}+\begin{bmatrix}2x_2+y_2\\x_2+3y_2\end{bmatrix}\end{cases}\quad (p.92)$$

$$\begin{cases}e_1=\begin{bmatrix}1\\0\end{bmatrix}=\vec{i}\\e_2=\begin{bmatrix}0\\1\end{bmatrix}=\vec{j}\end{cases}$$

證2)
$$\begin{cases}T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x+y\\x+3y\end{bmatrix}=\begin{bmatrix}2&1\\1&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=Ax\\(\text{或})=T(xe_1+ye_2)=xT(e_1)+yT(e_2)\end{cases}$$

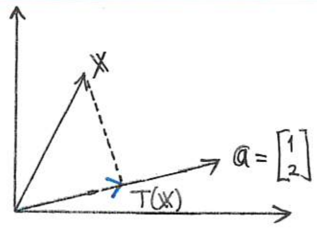
$$\begin{cases}T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}=a_1\\T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\3\end{bmatrix}=a_2\end{cases}$$

• **Example 2** (a)(b)(c) 中 (1) $T:\mathbb{R}^2\to\mathbb{R}^2$ 是線性, (2) $A=?$

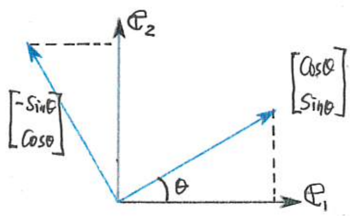
(a) $T(x)=\mathrm{proj}_a x=\dfrac{x\cdot a}{\|a\|^2}a=\dfrac{x+2y}{5}\begin{bmatrix}1\\2\end{bmatrix}=\dfrac{1}{5}\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=Ax$

(projection on a)

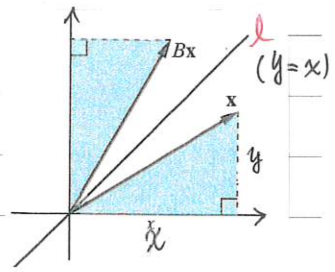
(或) (1) p.93, (2) $T(e_1)=\frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}$, $T(e_2)=\frac{1}{5}\begin{bmatrix}2\\4\end{bmatrix}$



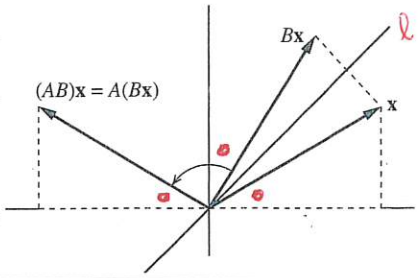
(b) $T(x)=$ rotate x by θ, $A_\theta=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$



(c) $\begin{cases} T_A : \text{rotate } \frac{\pi}{2}, & A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \\ T_B : \text{reflect acros } \ell, & B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \end{cases}$

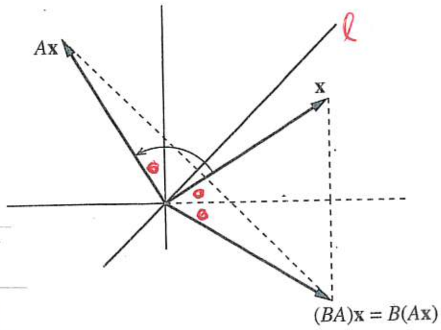


$\Rightarrow \begin{cases} AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; AB\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \\ BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \; BA\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \end{cases}$



• **Example 3** $T: \mathbb{R}^3 \to \mathbb{R}^3$ rotate θ around z-axis

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

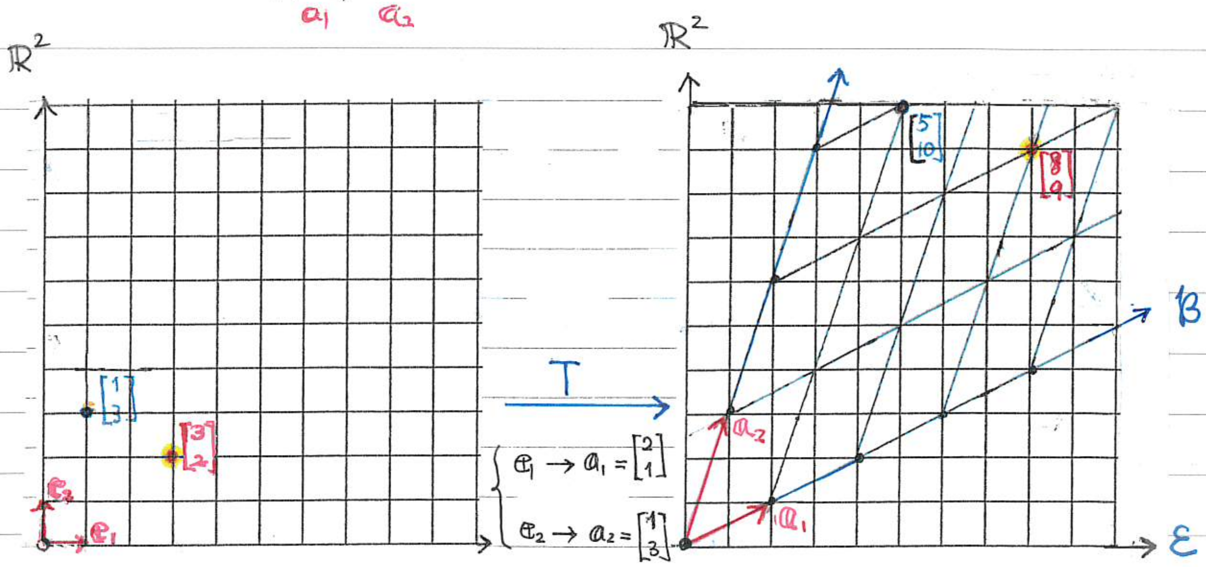


• **Remarks**

(A) 線性變換: $T(x) = \begin{bmatrix} 2x + y \\ x + 3y \end{bmatrix}$, $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = 3T(e_1) + 2T(e_2) = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$

(B) 矩陣: $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3a_1 + 2a_2 = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$

(C) 座標變換: $\begin{cases} \mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\ \mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{a_1}, \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{a_2} \right\} \end{cases}$, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_{\mathcal{B}} = 3a_1 + 2a_2 = \begin{bmatrix} 8 \\ 9 \end{bmatrix}_{\mathcal{E}}$



• **定理** $T: \mathbb{R}^2 \to \mathbb{R}^2$ 則 T: linear $\Leftrightarrow$ $T: \begin{cases} 0 \to 0 \\ \text{直線 } L \to \text{直線 } T(L) \end{cases}$
(conti, invertible)

證: "$\Rightarrow$" $L: x = x_0 + tu \Rightarrow T(x) = T(x_0) + t\,T(u)$

Elementary Matrices (row operations)

• $[x_1\ x_2\ x_3\ x_4\ x_5]\begin{bmatrix}A_1\\A_2\\A_3\\A_4\\A_5\end{bmatrix} = x_1A_1+\cdots+x_5A_5$

• $\boxed{\text{Elementary op}} \longleftrightarrow E_{m\times m} = \text{op } I_m$

• E^{-1} 存在，且形狀相同 $\longleftrightarrow$ op 反運算

• R_{25}: $\begin{bmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&1&0&0&0\end{bmatrix}\begin{bmatrix}A_1\\A_2\\A_3\\A_4\\A_5\end{bmatrix} = \begin{bmatrix}A_1\\A_5\\A_3\\A_4\\A_2\end{bmatrix}$ $\quad E^{-1} = E \quad : R_{25}$

• cR_4: $\begin{bmatrix}1&&&&\\&1&&&\\&&1&&\\&&&c&\\&&&&1\end{bmatrix}\begin{bmatrix}A_1\\A_2\\A_3\\A_4\\A_5\end{bmatrix} = \begin{bmatrix}A_1\\A_2\\A_3\\cA_4\\A_5\end{bmatrix}$ $\quad E^{-1} = \begin{bmatrix}1&&&&\\&1&&&\\&&1&&\\&&&\frac{1}{c}&\\&&&&1\end{bmatrix} : \frac{1}{c}R_4$

(下三角)

• cR_2+R_4: $\begin{bmatrix}1&&&&\\&1&&&\\&\vdots&1&&\\&c&\cdots&1&\\&&&&1\end{bmatrix}\begin{bmatrix}A_1\\A_2\\A_3\\A_4\\A_5\end{bmatrix} = \begin{bmatrix}A_1\\A_2\\A_3\\cA_2+A_4\\A_5\end{bmatrix}$ $\quad E^{-1} = \begin{bmatrix}1&&&&\\&1&&&\\&\vdots&1&&\\&-c&\cdots&1&\\&&&&1\end{bmatrix} : -cR_2+R_4$

• Example

$A = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$

R_{12}: $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix} = \begin{bmatrix}3&4\\1&2\\5&6\end{bmatrix}$ $\quad E_1^{-1} = E$

$4R_3$: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&4\end{bmatrix}\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix} = \begin{bmatrix}1&2\\3&4\\20&24\end{bmatrix}$ $\quad E_2^{-1} = \begin{bmatrix}1&&\\&1&\\&&\frac{1}{4}\end{bmatrix}$

$2R_1+R_3$: $\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix} = \begin{bmatrix}1&2\\3&4\\7&10\end{bmatrix}$ $\quad E_3^{-1} = \begin{bmatrix}1&&\\&1&\\-2&&1\end{bmatrix}$

• Example 解 $\begin{cases}4x+3y=5\\x+2y=5\end{cases}$, $\begin{bmatrix}4&3\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$, $A\mathbf{x} = \mathbf{b}$

解 $\left[\begin{array}{cc|c}4&3&5\\1&2&5\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1&2&5\\4&3&5\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1&2&5\\0&-5&-15\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1&2&5\\0&1&3\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1&0&-1\\0&1&3\end{array}\right]$, $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$

$E_1 = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\quad E_2 = \begin{bmatrix}1&0\\-4&1\end{bmatrix}$ $\quad E_3 = \begin{bmatrix}1&0\\0&-\frac{1}{5}\end{bmatrix}$ $\quad E_4 = \begin{bmatrix}1&-2\\0&1\end{bmatrix}$

$E = E_4E_3E_2E_1 = \begin{bmatrix}\frac{2}{5}&-\frac{3}{5}\\-\frac{1}{5}&\frac{4}{5}\end{bmatrix}$, $E_4E_3E_2E_1[A|\mathbf{b}] = E[A|\mathbf{b}] = \begin{bmatrix}\frac{2}{5}&-\frac{3}{5}\\-\frac{1}{5}&\frac{4}{5}\end{bmatrix}\left[\begin{array}{cc|c}4&3&5\\1&2&5\end{array}\right] = \left[\begin{array}{cc|c}1&0&-1\\0&1&3\end{array}\right]$

(全部計算過程)

• Remarks

(1) $E[A|\mathbf{b}] = [EA|E\mathbf{b}] = \left[\begin{array}{cc|c}1&0&-1\\0&1&3\end{array}\right]$ $\quad \begin{cases}EA = I_2\\E\mathbf{b} = \begin{bmatrix}-1\\3\end{bmatrix}\end{cases} \Rightarrow \begin{cases}E = A^{-1}\\\mathbf{x} = A^{-1}\mathbf{b} = E\mathbf{b} = \begin{bmatrix}-1\\3\end{bmatrix}\end{cases}$

(2) 如何求 $E\ (=A^{-1})$? $\quad \left[\begin{array}{cc|c}4&3&a\\1&2&b\end{array}\right] \xrightarrow{E} \left[\begin{array}{cc|c}1&0&\frac{2}{5}a-\frac{3}{5}b\\0&1&-\frac{1}{5}a+\frac{4}{5}b\end{array}\right]$ $\quad \therefore E = \begin{bmatrix}\frac{2}{5}&-\frac{3}{5}\\-\frac{1}{5}&\frac{4}{5}\end{bmatrix}$ (無需 E_i)

(Constraint eq.) $\quad \mathbf{b} \quad E\mathbf{b}$

Invertible Matrix 可逆矩陣, 反矩陣 (inverse)

No.
Date: / /

- **定義**: $A_{n\times n}$ 可逆 $\overset{def}{\Longleftrightarrow}$ 存在 E, $EA = AE = I_n$
  E (唯一)*: A之反矩陣 $(E = A^{-1})$

*(若 $FA = AF = I_n$ $\Rightarrow E = EAF = F$)

- **定理1** A, B 可逆 ⇒ (1) $A\mathbb{x} = \mathbb{b}$ 有唯一解 $\mathbb{x} = A^{-1}\mathbb{b}$
  (2) A^{-1} 可逆, $(A^{-1})^{-1} = A$ $(A^{-1} = E)$
  (3) AB 可逆, $(AB)^{-1} = B^{-1}A^{-1}$

- **定理2** A 可逆 ⇔ A nonsingular*
  證: "⇐" $A \overset{E_1}{\leadsto} A_1 \leadsto \cdots \overset{E_K}{\leadsto} I_n \Rightarrow E_K\cdots E_1 A = I_n$
  "⇒" [如推論(1)證] $\Rightarrow A = E_1^{-1}\cdots E_K^{-1}$ 可逆

* ⇔ $A \leadsto I_n$
  ⇔ $A\mathbb{x} = \mathbb{b}$ 有唯一解
  ⇔ $A\mathbb{x} = \mathbb{0}$ 只有0解
  ⇔ $\det(A) \neq 0$

- **推論** (1) 存在 E, $EA = I_n \Rightarrow$ A 可逆, $E = A^{-1}$
  (2) 〃 F, $AF = I_n \Rightarrow$ A 〃, $F = A^{-1}$

  證: (1) $\overset{E}{} A\mathbb{x} = \overset{E}{}\mathbb{0} \Rightarrow \mathbb{x} = \mathbb{0} \Rightarrow$ A可逆。$EA^{A^{-1}} = I_n^{A^{-1}}$
  (2) F 可逆, $F^{-1} = A$ 可逆 且 $A^{-1} = F$
  $(E = E_K\cdots E_1 = A^{-1})$

如何求 A^{-1}

$$AX = I_n$$

(甲) (解X)

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \overset{E}{\leadsto} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}$$

或 $[A \mid I_n] \overset{E}{\leadsto} [I_n \mid B] \Rightarrow \begin{cases} EA = I_n \\ EI_n = B \end{cases} \Rightarrow B = E = A^{-1}$

(乙) (Constraint Eq.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 2 & -1 & 0 & b \\ 1 & -2 & 2 & c \end{array}\right] \overset{E}{\leadsto} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2a - c \\ 0 & 1 & 0 & 4a - b - 2c \\ 0 & 0 & 1 & 3a - b - c \end{array}\right]$$

$[A \mid \mathbb{b}] \overset{E}{\leadsto} [EA \mid E\mathbb{b}]$, $A^{-1} = E = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}$

(丙) (Cramer ⇒ 餘因子矩陣)

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\det(A) = \Delta = -1$

$$\begin{cases} x_{11} = \dfrac{\begin{vmatrix} -1 & 0 \\ -2 & 2 \end{vmatrix}}{\Delta} = 2 \\ x_{21} = \dfrac{-\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}}{\Delta} = 4 \\ x_{31} = \dfrac{\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}}{\Delta} = 3 \end{cases}$$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

LU decomposition
(factorization)

(1) $A = LU$ $\begin{cases} L: \text{lower triangular} \\ U: \text{upper} \ \text{"} \end{cases}$

(2) $A\mathbf{x} = \mathbf{b}$

$L\underbrace{U\mathbf{x}}_{\mathbf{y}} = \mathbf{b} \Rightarrow \begin{cases} (甲)\ 先解\ \ L\mathbf{y} = \mathbf{b} & (\text{forward substitution}) \\ (乙)\ 再解\ \ U\mathbf{x} = \mathbf{y} & (\text{backward} \ \text{"} \ ) \end{cases}$

- Example

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 7 & 4 & 2 \\ -1 & 4 & 13 & -1 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ -1 & 4 & 13 & -1 \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 6 & 12 & 0 \end{bmatrix} \xrightarrow{(-2)R_2+R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$E_1 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$$

$E_3E_2E_1A = U$ (no R_{ij})

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= LU$

解 $A\mathbf{x} = \mathbf{b}$, $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 7 & 4 & 2 \\ -1 & 4 & 13 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$L\underbrace{U\mathbf{x}}_{\mathbf{y}} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{y}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\Downarrow$

$$\begin{cases} (甲) \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \quad \begin{cases} y_1 = 1 \\ 2y_1 + y_2 = 2 \\ -y_1 + 2y_2 + y_3 = 3 \end{cases} \\ (乙) \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \end{cases}$$

$\boxed{\text{Transpose}}$ (轉置矩陣)

- $A^T = [a_{ij}]^T_{m\times n} = [b_{ij}]_{n\times m},\ b_{ij} = a_{ji},$ $\begin{cases} A \quad \text{symmetric}: & A^T = A \\ A: \text{skew-symmetric}: & A^T = -A \end{cases}$

(註) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [a\ b\ c]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax+by+cz$

$\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y}$

- **Property**

(1) $(A^T)^T = A$ (4) $(AB)^T = B^TA^T$ $(m\times n)(n\times p) = (m\times p)$

(2) $(A+B)^T = A^T + B^T$ (5) $(A^T)^{-1} = (A^{-1})^T$

(3) $(cA)^T = cA^T$

Pf. (4) 32-entry: $L = A_2\cdot \mathbf{b}_3 = \mathbf{b}_3\cdot A_2 = R$

(5) $(AA^{-1})^T = I_n^T$

$(A^{-1})^TA^T = I_n$

- **Proposition** (Cross the dot) $A_{m\times n}: \mathbb{R}^n \to \mathbb{R}^m,\ \begin{cases} \mathbf{x}\in\mathbb{R}^n \\ \mathbf{y}\in\mathbb{R}^m \end{cases} \Rightarrow (A\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A^T\mathbf{y})$

Pf. $(A\mathbf{x})\cdot\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^TA^T\mathbf{y} = \mathbf{x}\cdot(A^T\mathbf{y})$ $\begin{cases} (1)\ (A^T\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y}) \\ (2)\ \langle A\mathbf{x},\mathbf{y}\rangle = \langle\mathbf{x},A^T\mathbf{y}\rangle \end{cases}$

- **Example**

(玩具) $\boxed{甲\quad 乙\quad 丙}$

(零件)

甲乙丙 生產量; ABCD 需要量; ABCD 單位價格

(1) $(A\mathbf{x})\cdot\mathbf{y}:$ $\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} \text{A}\ y_1\to \\ \text{B}\ y_2\to \\ \text{C}\ y_3\to \\ \text{D}\ y_4\to \end{array} & \begin{bmatrix} 3 & 2 & 4 \\ 5 & 7 & 3 \\ 4 & 6 & 2 \\ 7 & 4 & 5 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\cdot\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = T$ (總生產成本)

$= \sum_{i,j} a_{ij}y_ix_j$

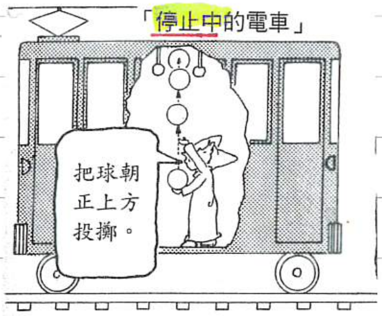
(2) $\mathbf{x}\cdot(A^T\mathbf{y}):$ $\begin{array}{cc} \begin{array}{c} 甲\to \\ 乙\to \\ 丙\to \end{array} & \begin{bmatrix} 3 & 5 & 4 & 7 \\ 2 & 7 & 6 & 4 \\ 4 & 3 & 2 & 5 \end{bmatrix} \\ & \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{A} & \text{B} & \text{C} & \text{D} \end{array} \end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T$

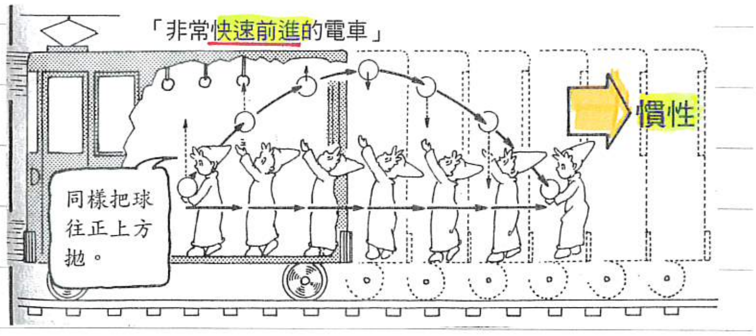
甲乙丙 單位生產成本

Special Relativity (特殊相對論)

物理原理

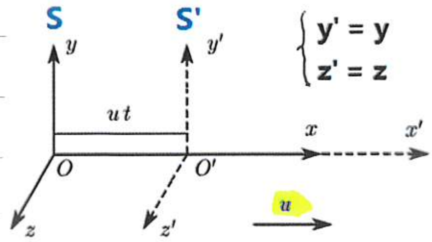
$\begin{cases}\text{(P1) 光速恆定原理：任何情況下，光速不變，恆為 } C=30\text{萬公里/秒 (1光秒/秒)。} \\ \text{(P2) 相對性原理：任何慣性系統(等速直線運動)中，物理定律均相同。}\end{cases}$





場景

$\begin{cases}\text{(A1) 電車(慣性系統 } S'(x',t')\text{) 以等速 } u \text{ 沿 } x \text{ 軸正方向移動} \\ \text{(A2) 地面(靜止系統 } S(x,t)\text{) 滿足 } S'(0,0)=S(0,0)\end{cases}$



$\begin{cases} y'=y \\ z'=z \end{cases}$

定理 (Lorentz 變換 L_u)

(P2): $L_u^{-1} = L(-u)$

$$\begin{cases} x' = \dfrac{x-ut}{\sqrt{1-u^2/C^2}} \\ t' = \dfrac{t-\frac{u}{C^2}x}{\sqrt{1-u^2/C^2}} \end{cases} \quad \text{即} \quad \begin{cases} \begin{bmatrix} x' \\ t' \end{bmatrix} = \dfrac{1}{\sqrt{1-u^2/C^2}}\begin{bmatrix} 1 & -u \\ \frac{-u}{C^2} & 1 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} \overset{C=1}{=} \dfrac{1}{\sqrt{1-u^2}}\begin{bmatrix} 1 & -u \\ -u & 1 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} \overset{u=\frac{3}{5}}{=} \begin{bmatrix} \frac{5}{4} & \frac{-3}{4} \\ \frac{-3}{4} & \frac{5}{4} \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} \\ \begin{bmatrix} x \\ t \end{bmatrix} = \underbrace{\dfrac{1}{\sqrt{1-u^2/C^2}}\begin{bmatrix} 1 & u \\ \frac{u}{C^2} & 1 \end{bmatrix}}_{L_u}\begin{bmatrix} x' \\ t' \end{bmatrix} = \dfrac{1}{\sqrt{1-u^2}}\begin{bmatrix} 1 & u \\ u & 1 \end{bmatrix}\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}\begin{bmatrix} x' \\ t' \end{bmatrix} \end{cases}$$

$(u=\frac{3}{5},\ \alpha=\sqrt{1-u^2}=\frac{4}{5})$

証

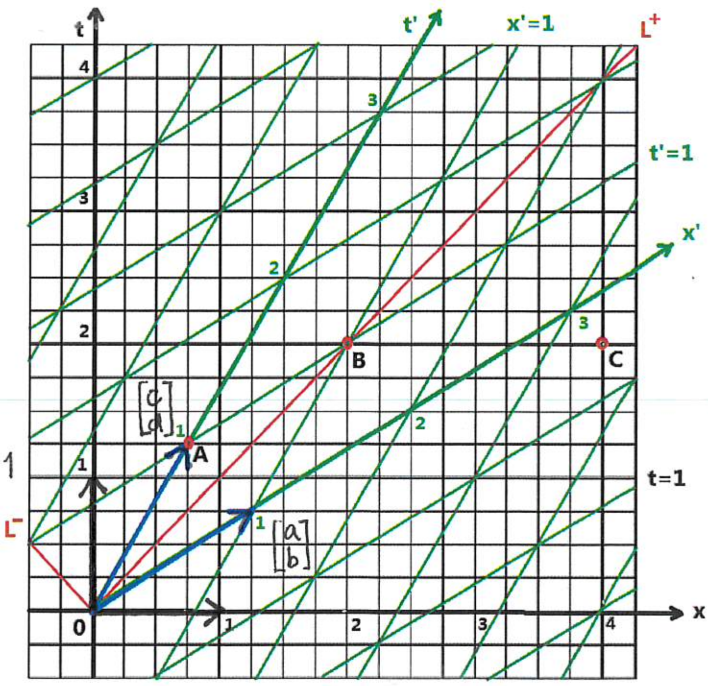
(P0) L_u: 線性變換，令 $L_u=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $\quad c=du$

$\because \left(\begin{matrix}\text{直線} \\ \text{(等速直線運動)}\end{matrix} \longrightarrow \begin{matrix}\text{直線} \\ \text{(等速直線運動)}\end{matrix}\right)^*$

(P1) $\begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix}', \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in L^+ \\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}', \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in L^- \end{cases} \Rightarrow \begin{cases} L_u\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ ^{**} \\ L_u\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$

$\begin{cases} a+c = b+d \\ -(-a+c) = -b+d \end{cases} \Rightarrow \begin{cases} a=d \\ b=c \end{cases}, \quad L_u = d\begin{bmatrix} 1 & u \\ u & 1 \end{bmatrix}$

(P2) $L_u L(-u) = d^2\begin{bmatrix} 1 & u \\ u & 1 \end{bmatrix}\begin{bmatrix} 1 & -u \\ -u & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow d^2(1-u^2)=1$



物理現象

(A) 同時性：

事件	$S'(x',t')$	$S(x,t)$
A	$(0,1)$	$(\frac{3}{4},\frac{5}{4})$
B	$(1,1)$	$(2,2)$
C	$(\frac{7}{2},\frac{-1}{2})$	$(4,2)$

$\therefore$ A,B 在 S' 同時，在 S 不同時／BC 在 S 同時，在 S' 則否

(*) Einstein：時空是均勻的， (**) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ 是 L_u 的 eigen value.

$$\begin{bmatrix}x'\\t'\end{bmatrix}\xrightarrow{L_u=\frac{1}{\alpha}\begin{bmatrix}1&u\\u&1\end{bmatrix}\overset{u=\frac{3}{5}}{=\!=}\begin{bmatrix}5/4&3/4\\3/4&5/4\end{bmatrix}}\begin{bmatrix}x\\t\end{bmatrix}$$

$$L_u^{-1}=\frac{1}{\alpha}\begin{bmatrix}1&-u\\-u&1\end{bmatrix}=\begin{bmatrix}5/4&-3/4\\-3/4&5/4\end{bmatrix}$$

$\alpha=\sqrt{1-u^2}\ \overset{u=\frac{3}{5}}{=\!=}\ \frac{4}{5}$
(收縮因子)

(B) 時間膨脹： $\Delta t'=\alpha\,\Delta t$ （固有時間最短）

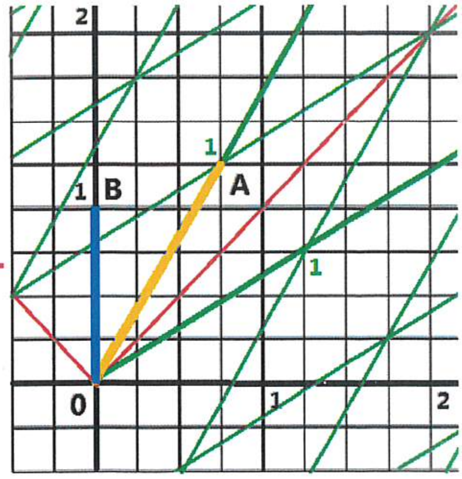
事件	$S'(x',t')$	$S(x,t)$
O	$(0,0)$	$(0,0)$
A	$(0,1)$	$(\frac{3}{4},\frac{5}{4})$
B	$(\frac{-3}{4},\frac{5}{4})$	$(0,1)$
X	$(0,t')$	$(\frac{ut'}{\alpha},\frac{t'}{\alpha})$
Y	$(\frac{-ut}{\alpha},\frac{t}{\alpha})$	$(0,t)$

(O: 燈亮, A: 燈熄)

$\overline{OA}$ $\begin{cases}\Delta t'=1 & \text{(固有時間：同地測量)}\\ \Delta t=\frac{5}{4} & \alpha=\frac{4}{5}\end{cases}$

$\overline{OB}$ $\begin{cases}\Delta t'=\frac{5}{4}\\ \Delta t=1 & \text{(固有時間)}\end{cases}$

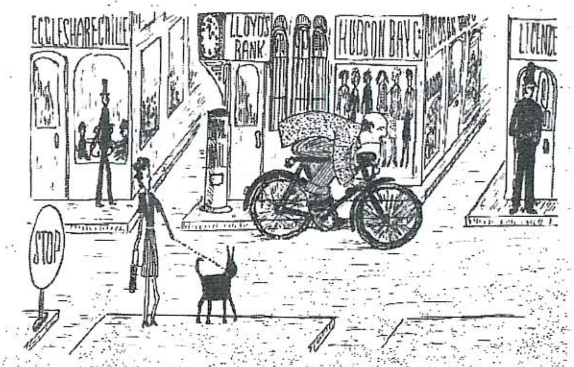
$\overline{OX}$ $\begin{cases}\Delta t'=t' & \text{(固有時間)}\\ \Delta t=\frac{t'}{\alpha}\end{cases}$



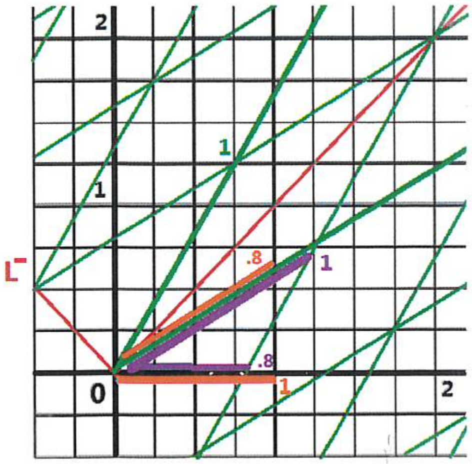
(C) 長度收縮： $L=\alpha L'$

(動尺) 運動中電車： $L'=1,\ L=\frac{4}{5}$

地面靜止之尺： $\ell'=\frac{4}{5}\quad \ell=1$



The city blocks became still shorter



• 定理（速度加成）

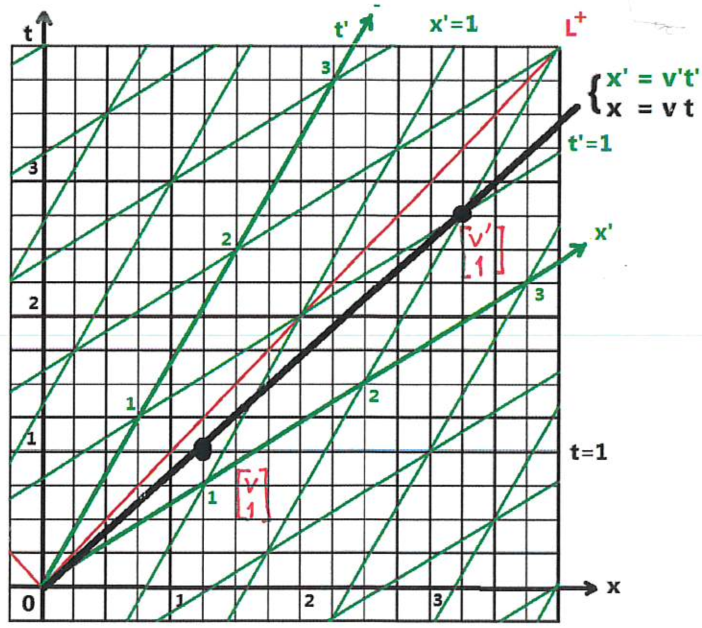
電車中射出速度為 v' 之子彈，
則地面觀測子彈速度 $v=\dfrac{u+v'}{1+\frac{uv'}{c^2}}$

證

$$\begin{bmatrix}v'\\1\end{bmatrix}\xrightarrow{L_u}\frac{1}{\alpha}\begin{bmatrix}v'+u\\uv'+1\end{bmatrix}\ /\!/\ \begin{bmatrix}\frac{u+v'}{1+uv'}\\1\end{bmatrix}=\begin{bmatrix}v\\1\end{bmatrix}$$

$(c=1)$

註：(1) $u, v'\ll c$, $v\doteqdot u+v'$
(2) u 或 $v'=c$, $v=c$



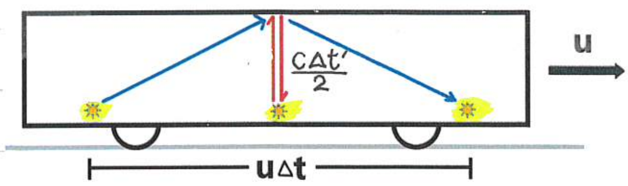
* George Gamov: Mr. Tompkins in Wonderland

物理教科書之推導

• 定理1

(A) (時間膨脹) $\Delta t' = \sqrt{1-\frac{u^2}{c^2}}\,\Delta t$

(B) (長度收縮) $L = \sqrt{1-\frac{u^2}{c^2}}\,L'$

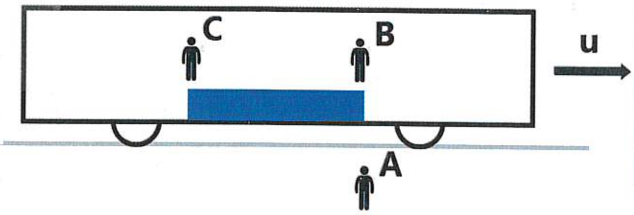


証:

(A) $\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{c\Delta t'}{2}\right)^2 + \left(\frac{u\Delta t}{2}\right)^2 \Rightarrow \Delta t^2 = (\Delta t')^2 + \left(\frac{u}{c}\right)^2 \Delta t^2$ ($\Delta t'$: 固有時間, 同地觀測)

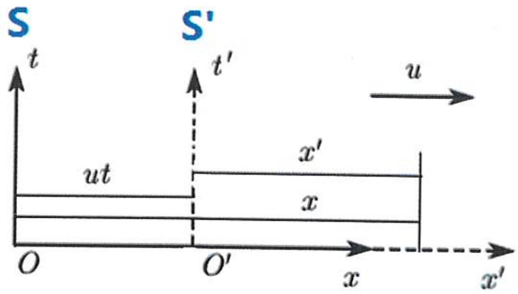
($\alpha = \sqrt{1-\frac{u^2}{c^2}}$: 收縮因子)

(B) $\begin{cases} L = u\Delta t \\ L' = u\Delta t' \end{cases}$ (Δt: 固有時間 $\Rightarrow \Delta t = \alpha \Delta t'$)



• 定理2 (Lorentz 變換)

$$\begin{cases} x' = \dfrac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}} \\ t' = \dfrac{t-\frac{u}{c^2}x}{\sqrt{1-\frac{u^2}{c^2}}} \end{cases}$$



証: $\begin{cases} (P1) & x-ut = \alpha x' \\ (P2) & x'+ut' = \alpha x \end{cases} \Rightarrow \alpha u t' = (\alpha^2-1)x + ut$

• 定理3 (速度加成)

$V = \dfrac{u+V'}{1+\frac{uV'}{c^2}}$ ($V' = \frac{dx'}{dt'}$)

証: $V = \dfrac{dx}{dt} = \dfrac{\frac{1}{\alpha}(dx'+u\,dt')}{\frac{1}{\alpha}\left(\frac{u}{c^2}dx'+dt'\right)} = \dfrac{\frac{dx'}{dt'}+u}{\frac{u}{c^2}\frac{dx'}{dt'}+1}$