Chapter 3 Vector Spaces

綱要　向量空間 $(\mathbb{R}^n,+,\cdot)$

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(A) 子空間 (subspaces)

- $V \lhd \mathbb{R}^n$，當 (1) $\begin{cases} V\subset\mathbb{R}^n \\ (V,+,\cdot)\text{ 是向量空間} \end{cases} \Leftrightarrow$ (2) $\begin{cases} u,v\in V \\ c,d\in\mathbb{R} \end{cases}$ 則 $\begin{matrix} u+v\in V \\ cv\in V \end{matrix} \Leftrightarrow$ (3) $cu+dv\in V$

- $V=\begin{cases} \text{Span}(v_1,\dots,v_k) & \text{(參數式)} \\ \{y\in\mathbb{R}^n \mid Ay=0\} & \text{(方程式)} \end{cases} \lhd \mathbb{R}^n$，$\begin{matrix} \dim V = k & \text{if } \{v_1,\dots,v_k\}\text{ 線性獨立} \\ \dim V = n-k & \text{if } \text{rank}(A)=k. \end{matrix}$

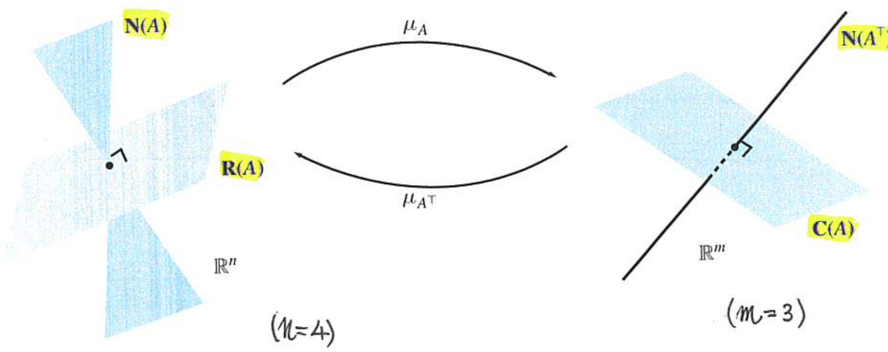
  (平面物件：$x_0+V = x_0+\text{Span}(v_1,\dots,v_k) = \{x\in\mathbb{R}^n \mid Ax=b\}$　$(Ax_0=b)$)

- $U, V \lhd \mathbb{R}^n \Rightarrow U\cap V,\ V^\perp,\ U\oplus V \lhd \mathbb{R}^n$

- $\boxed{\text{四大子空間}}$　$A\in M_{m\times n}$，$T_A:\mathbb{R}^n\to\mathbb{R}^m$

  $R(A)=\text{Span}(A_1,\dots,A_m) \lhd \mathbb{R}^n$，　$N(A)=\{x\in\mathbb{R}^n \mid Ax=T_A(x)=0\}=\text{Ker}(T)\lhd\mathbb{R}^n$

  $C(A)=\text{Span}(a_1,\dots,a_n)=\text{Im}(T)\lhd\mathbb{R}^m$，　$N(A^T)=\{y\in\mathbb{R}^m \mid A^Ty=0\}\lhd\mathbb{R}^m$



$(n=4)$　　　　$(m=3)$

(B) 基底 (basis)，線性獨立 (linear independent)

(1) $\{v_1,\dots,v_k\}$ 是線性獨立，當 $c_1v_1+\dots+c_kv_k=0 \Rightarrow \begin{bmatrix} c_1\\ \vdots \\ c_k\end{bmatrix}=0$　(c_i 全為 0)

(2) $\{v_1,\dots,v_k\}$：線性相依，$\exists \begin{bmatrix} c_1\\ \vdots \\ c_k\end{bmatrix}\neq 0$，$\ni c_1v_1+\dots+c_kv_k=0$　(不全為 0)

(3) $\{v_1,\dots,v_k\}$ 是 V 的基底，當 $\begin{cases} (1)\ \text{Span}(v_1,\dots,v_k)=V, \\ (2)\ \{v_1,\dots,v_k\}\text{ 線性獨立}, \end{cases}$　$\dim V = k$

(4) $\boxed{\text{線性代數基本定理}}$

$\begin{cases} R(A)^\perp = N(A), & \mathbb{R}^n = R(A)\oplus N(A), \\ C(A)^\perp = N(A^T), & \mathbb{R}^m = C(A)\oplus N(A^T), \end{cases}$　$\dim R(A)=\dim C(A)=\text{rank}(A)$

(C) 抽象向量空間

(1) $(V,+,\cdot)$：$\mathbb{R}^n$，$\mathbb{R}^\omega$，$\mathcal{F}([a,b])$，$M_{m\times n}$，$L(V,W)$

(2) $L(U,V)=\{T: U\to V \mid T\text{ 線性}\}$

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- **定義** $\begin{cases} (\mathbb{R}^n, +, \cdot) \text{ 向量空間} \\ V \subset \mathbb{R}^n,\ (V, +, \cdot) \text{ 向量空間} \end{cases}$ 則 $\begin{cases} V \lhd \mathbb{R}^n \\ (\mathbb{R}^n \text{ 的子空間}) \end{cases}$

- **Proposition**

(1) $V \lhd \mathbb{R}^n \iff \begin{cases} \text{(i) } 0 \\ \text{(ii) } u+v \\ \text{(iii) } cv \end{cases} \in V,\quad \forall \begin{cases} u, v \in V \\ c, d \in \mathbb{R} \end{cases} \iff cu + dv \in V$

(2) $\mathrm{Span}(v_1, \cdots, v_k) \lhd \mathbb{R}^n$

註: $\{0\} \lhd \mathrm{Span}(v_1) \lhd \mathrm{Span}(v_1, v_2) \lhd \cdots \lhd \mathrm{Span}(v_1, \cdots, v_k)$

(3) $\{x \in \mathbb{R}^n \mid Ax = 0\} \lhd \mathbb{R}^n$

證:

(2) $\begin{cases} (c_1v_1 + \cdots + c_kv_k) + (d_1v_1 + \cdots + d_kv_k) = (c_1+d_1)v_1 + \cdots + (c_k+d_k)v_k \\ c(c_1v_1 + \cdots + c_kv_k) = cc_1v_1 + \cdots + cc_kv_k \end{cases}$

(3) $\begin{cases} A(x+y) = Ax + Ay = 0 \\ A(cx) = cAx = 0 \end{cases}$

- **Exercise** $S_1, S_2, S_3 \not\lhd \mathbb{R}^2$

不成立

(1) $S_1 = \{(x,y) \mid y = 2x+1\}$ (i)(ii)(iii) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S_1$

(2) $S_2 = \{(x,y) \mid xy = 0\}$ (ii) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S_2$

(3) $S_3 = \{(x,y) \mid y \geq 0\}$ (iii) $(-2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \notin S_3$

- **Example** $\mathbb{R}^3$

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\quad v = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$w = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},\quad x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = w - v$

$P_1 = \mathrm{Span}(v, w) \lhd \mathbb{R}^3$

$P_2 = \{u + sv + tw \mid s, t \in \mathbb{R}\} \not\lhd \mathbb{R}^3$

$P_3 = \{x + sv + tw \mid \ \text{〃}\ \} \lhd \mathbb{R}^3$

說明: $0 \notin P_2$, $\because \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (s,t 存在?)

$\begin{cases} s + 2t + 1 = 0 \\ -s = 0 \\ 2s + t = 0 \end{cases}$ (無解)

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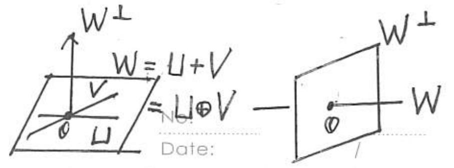
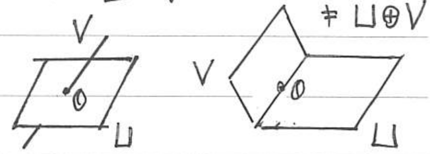


$\mathbb{R}^3 = U\oplus V$ $\qquad$ $\mathbb{R}^3 = U+V \neq U\oplus V$



- **定義** $\begin{cases} (1)\ \mathbf{u}\perp V := \mathbf{u}\perp\mathbf{v},\ \forall\, \mathbf{v}\in V \\ (2)\ U\perp V := \mathbf{u}\perp\mathbf{v},\ \forall\, \mathbf{u}\in U,\ \mathbf{v}\in V \end{cases}$

- **定義** $\begin{cases} U \\ V \end{cases} \triangleleft\ \mathbb{R}^n,\ \begin{cases} U+V = \{\mathbf{u}+\mathbf{v} \mid \mathbf{u}\in U, \mathbf{v}\in V\} & \text{sum of } U, V \quad (V \text{ prep}) \\ V^\perp = \{\mathbf{x}\in\mathbb{R}^n \mid \mathbf{x}\perp V\} & \text{orthogonal complement of } V \\ W = U\oplus V \Leftrightarrow \begin{cases} W = U+V \\ U\cap V = \{\mathbf{0}\} \end{cases} & \text{direct sum of } U, V \end{cases}$

- **定理** $\begin{cases} U \\ V \end{cases} \triangleleft\ \mathbb{R}^n \Rightarrow \begin{cases} U\cap V \triangleleft\ \mathbb{R}^n \\ U+V \triangleleft\ \mathbb{R}^n \\ V^\perp \triangleleft\ \mathbb{R}^n \end{cases}$

  證 $\begin{cases} \mathbf{u} \\ \mathbf{v} \end{cases} \in \_\_\_\_ \Rightarrow \begin{cases} \mathbf{u}+\mathbf{v} \\ c\mathbf{u} \end{cases} \in \_\_\_\_$

  $U\cap V,\ U+V,\ V^\perp$

  $(Q: U\cup V\ ?)$

- **定理** (1) $V \perp V^\perp$ $\qquad$ (V的法空間)

  (2) $V \subset (V^\perp)^\perp$ $\qquad$ (定理4.9: $V = (V^\perp)^\perp$, $\mathbb{R}^n = V\oplus V^\perp$)

  (3) $V\cap V^\perp = \{\mathbf{0}\}$ $\qquad$ ($\mathbf{v}\cdot\mathbf{v} = 0 \Rightarrow \mathbf{v} = \mathbf{0}$)

- **Example** $\begin{cases} P_1 = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right) \\ P_2 = \text{span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right) \end{cases}$ 則 $P_1\cap P_2 = \text{span}\left(\begin{bmatrix}1\\2\\2\end{bmatrix}\right) \triangleleft\ \mathbb{R}^3$

  證 $\mathbf{x} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} = c\begin{bmatrix}-1\\1\\0\end{bmatrix} + d\begin{bmatrix}2\\1\\2\end{bmatrix} \in P_1\cap P_2 \Rightarrow \mathbf{x} = d\begin{bmatrix}1\\2\\2\end{bmatrix}$

  $\because \begin{bmatrix}1&0&1&-2\\0&1&-1&-1\\0&1&0&-2\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},\ \begin{bmatrix} & \end{bmatrix} \rightsquigarrow \begin{bmatrix}1&0&0&-1\\0&1&0&-2\\0&0&1&-1\end{bmatrix} \Rightarrow \begin{bmatrix}a\\b\\c\\d\end{bmatrix} = d\begin{bmatrix}1\\2\\1\\1\end{bmatrix}$

- **Example** $V = \text{span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right)$, $V^\perp = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \,\middle|\, x+2y+z=0\right\} = W$, 試證 $W^\perp = V$

  證: "$\supset$"

  "$\subset$": $W = \text{span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$, $\because \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2y-z\\y\\z\end{bmatrix} = y\begin{bmatrix}-2\\1\\0\end{bmatrix} + z\begin{bmatrix}-1\\0\\1\end{bmatrix}$

  $W^\perp = V$, $\because \begin{bmatrix}x\\y\\z\end{bmatrix} \in W^\perp \Leftrightarrow \begin{cases} -2x+y = 0 \\ -x+z = 0 \end{cases} \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\2x\\x\end{bmatrix} = x\begin{bmatrix}1\\2\\1\end{bmatrix}$

• **定理** $\begin{cases} \mathbb{U} \triangleleft \mathbb{R}^n \\ \mathbb{V} \triangleleft \mathbb{R}^n \end{cases}$, $\mathbb{W} = \mathbb{U} + \mathbb{V} = \left\{ u+v \;\middle|\; \begin{matrix} u \in \mathbb{U} \\ v \in \mathbb{V} \end{matrix} \right\}$，則下列敘述等價 $(\mathbb{W} = \mathbb{U} \oplus \mathbb{V})$

(a) $\mathbb{U} \cap \mathbb{V} = \{0\}$

(b) $\begin{cases} u \in \mathbb{U} \\ v \in \mathbb{V} \end{cases}$, $u + v = 0 \Rightarrow \begin{cases} u = 0 \\ v = 0 \end{cases}$

(c) $w \in \mathbb{W}$，則 $w = u + v$ (唯一)，其中 $\begin{cases} u \in \mathbb{U} \\ v \in \mathbb{V} \end{cases}$

(d) $\begin{cases} \mathcal{B}_1 = \{u_1, \dots, u_p\} \\ \mathcal{B}_2 = \{v_1, \dots, v_q\} \end{cases}$ basis for $\begin{matrix} \mathbb{U} \\ \mathbb{V} \end{matrix}$，則 $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ basis for $\mathbb{W}$

証 (a)⇒(b)：$u + v = 0 \Rightarrow u = -v \in \mathbb{U} \cap \mathbb{V} \Rightarrow u = -v = 0$

(b)⇒(c)：$w = u_1 + v_1 = u_2 + v_2 \Rightarrow (u_1 - u_2) + (v_1 - v_2) = 0 \Rightarrow \begin{cases} u_1 - u_2 = 0 \\ v_1 - v_2 = 0 \end{cases}$

(c)⇒(a)：$x\,(\neq 0) \in \mathbb{U} \cap \mathbb{V}$, $x = x + 0 = 0 + x$ $\times$

(c)⇔(d)：$w \in \mathbb{W}$, $w = \underbrace{u + v}_{(u \in \mathbb{U},\, v \in \mathbb{V} \text{ 唯一})} = \underbrace{c_1 u_1 + \cdots + c_p u_p + d_1 v_1 + \cdots + d_q v_q}_{(u,\ v \text{ 唯一})} \Leftrightarrow \mathcal{B}$ basis for $\mathbb{W}$

(註：$\dim \mathbb{W} = \dim(\mathbb{U} \oplus \mathbb{V}) = \dim \mathbb{U} + \dim \mathbb{V}$[*])

• **Proposition** $T: \mathbb{R}^n \to \mathbb{R}^m$ 線性變換

(1) $\mathrm{Ker}(T) = \{ x \in \mathbb{R}^n \mid T(x) = 0 \} \triangleleft \mathbb{R}^n$

(2) $\mathrm{Im}(T) = \{ T(x) \mid x \in \mathbb{R}^n \} \triangleleft \mathbb{R}^m$

(3) T: 1-1 $\Leftrightarrow \mathrm{Ker}(T) = \{0\}$

証 (1) $u, v \in \mathrm{Ker}(T)$, $T(u) = T(v) = 0 \Rightarrow \begin{cases} T(u+v) = 0 \\ T(cu) = 0 \end{cases} \Rightarrow \begin{cases} u+v \\ cu \end{cases} \in \mathrm{Ker}(T)$

(2) $\begin{cases} u \\ v \end{cases} \in \mathrm{Im}(T)$, $\begin{matrix} T(x) = u \\ T(y) = v \end{matrix} \Rightarrow \begin{cases} T(x+y) = u+v \\ T(cx) = cu \end{cases} \Rightarrow \begin{cases} u+v \\ cu \end{cases} \in \mathrm{Im}(T)$

(3) $\begin{cases} \text{“}\Rightarrow\text{”} & T(0) = 0 \\ \text{“}\Leftarrow\text{”} & \underline{T(x) = T(y)} \Rightarrow \underline{T(x-y) = 0} \Rightarrow x - y = 0 \Rightarrow \underline{x = y} \end{cases}$

• **Proposition** 基本列運算保持 $\mathrm{Span}(\cdot)$

$\begin{cases} R_{13} : \mathrm{Span}(A, B, C, D) = \mathrm{Span}(C, B, A, D) \\ 5R_2 : \mathrm{Span}(A, B, C, D) = \mathrm{Span}(A, 5B, C, D) \\ 5R_2 + R_4 : \mathrm{Span}(A, B, C, D) = \mathrm{Span}(A, B, C, 5B+D) \end{cases}$

$aA + bB + cC + d(5B+D)$
$= aA + (b+5d)B + cC + dD$

• **Proposition**

$x \perp \mathrm{Span}(v_1, \dots, v_k) \Leftrightarrow x \perp v_i,\ (1 \le i \le k)$

(*) 定理：$\dim(\mathbb{U}+\mathbb{V}) = \dim \mathbb{U} + \dim \mathbb{V} - \dim(\mathbb{U} \cap \mathbb{V})$

Fundamental Theorem of Linear Algebra

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Date: / /

(四大子空間)

• 定義 (Four fundamental subspaces) $A_{m\times n}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T = \mu_A \quad x \longmapsto b = Ax$

(1) (row space) $R(A) = \mathrm{Span}(A_1, \cdots, A_m)$

(2) (column 〃) $C(A) = \mathrm{Span}(a_1, \cdots, a_n) = \{Ax \mid x \in \mathbb{R}^n\} = \{T(x) \mid x \in \mathbb{R}^n\} = \mathrm{Im}(T)$

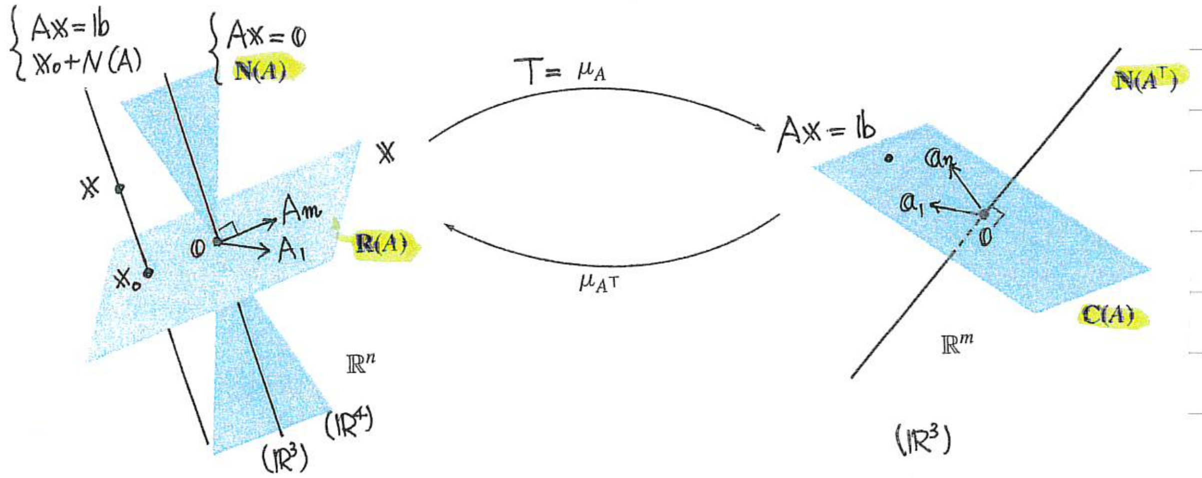
$= \{x_1 a_1 + \cdots + x_n a_n \mid x_i \in \mathbb{R}\} = R(A^T)$

(3) (null 〃) $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \{x \in \mathbb{R}^n \mid T(x) = 0\} = \mathrm{Ker}(T)$

$= \{x \in \mathbb{R}^n \mid x \perp A_1, \cdots, x \perp A_m\}$

(4) (left null 〃) $N(A^T) = \{y \in \mathbb{R}^m \mid A^T y = 0\} = \{y \in \mathbb{R}^m \mid y \perp a_1, \cdots, y \perp a_n\}$

• Proposition $\begin{cases} R(A),\ N(A) \lhd \mathbb{R}^n \\ C(A),\ N(A^T) \lhd \mathbb{R}^m \end{cases}$



• 定理 4.10 (線性代數基本定理)

(1) $\begin{cases} R(A)^\perp = N(A), & N(A)^\perp = R(A) \\ C(A)^\perp = N(A^T), & N(A^T)^\perp = C(A) \end{cases}$

(2)* $\begin{cases} \dim R(A) = \dim C(A) = r = \mathrm{rank}(A) \\ \dim N(A) = n - r \\ \dim N(A^T) = m - r \end{cases}$ *(後証)

(3)* $\begin{cases} \mathbb{R}^n = R(A) \oplus N(A) \\ \mathbb{R}^m = C(A) \oplus N(A^T) \end{cases}$

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• **Example 1** (pp.137) $A=\begin{bmatrix}1&-1&1&2\\1&0&1&1\\1&2&1&-1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&0&1&1\\0&1&0&-1\\0&0&0&0\end{bmatrix}$, $T_A:\mathbb{R}^4\longrightarrow\mathbb{R}^3$, $r=\text{rank}(A)=2$

求

• $R(A) = \text{Span}(A_1, A_2, A_3) = \text{span}\left(\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\right)$

• $N(A) = \{\mathbf{x}\in\mathbb{R}^4 \mid A\mathbf{x}=\mathbf{0}\} = \text{Span}\left(\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right)$, $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-x_3-x_4\\x_4\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}$

• $C(A) = \text{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = R(A^T) = \text{span}\left(\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right)$
$= \{\mathbf{b}\in\mathbb{R}^3 \mid A\mathbf{x}=\mathbf{b}\}$

(甲) $R(A^T)$ (乙) Const. Eq.
(丙) 課本標準算法: pp152, 163
$C(A) = \text{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}\right)$

• $N(A^T) = \{\mathbf{x}\in\mathbb{R}^3 \mid A^T\mathbf{x}=\mathbf{0}\} = \text{span}\left(\begin{bmatrix}2\\-3\\1\end{bmatrix}\right)$ (或課本標準算法)

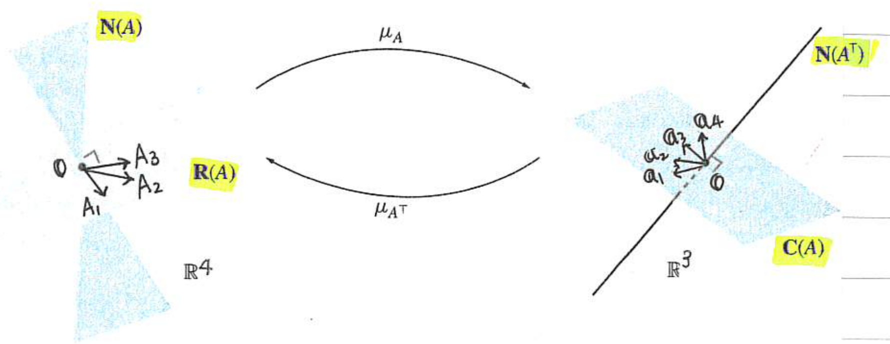
(乙) 求 $C(A)$ (參數式 $\xrightarrow[\text{(a)}]{\text{Const. Eq.}}$ 方程式 $\xrightarrow[\text{(b)}]{\text{高斯消去法}}$ 參數式)

(a) $C(A) = \left\{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \,\middle|\, A\mathbf{x}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}\right\}$ $\left[\begin{array}{cccc|c}1&-1&1&2&b_1\\1&0&1&1&b_2\\1&2&1&1&b_3\end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c}1&0&1&1&b_1\\0&1&0&-1&-b_1+b_2\\0&0&0&0&2b_1-3b_2+b_3\end{array}\right]$

$= \left\{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \,\middle|\, 2b_1-3b_2+b_3=0\right\}$ $\Rightarrow \begin{bmatrix}2\\-3\\1\end{bmatrix}\in N(A^T)$

(b) $= \text{span}\left(\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right)$ $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\-2b_1+3b_2\end{bmatrix}=b_1\begin{bmatrix}1\\0\\-2\end{bmatrix}+b_2\begin{bmatrix}0\\1\\3\end{bmatrix}$

• $\begin{cases}\dim R(A) = \dim C(A) = 2 = r\\ \dim N(A) = 2 \quad = n-r\\ \dim N(A^T) = 1 \quad = m-r\end{cases}$



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- Example 2 (pp.139) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$, $r = rank(A) = 2$

求

- $R(A) = span(A_1, A_2, A_3, A_4) = span(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$
- $N(A) = \{ \mathbb{x} \in \mathbb{R}^2 \mid A\mathbb{x} = \mathbb{0} \} = \{ \mathbb{0} \}$ $\quad \begin{cases} x_1 + 2x_2 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $C(A) = span(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix})$ or $span(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix})$
- $N(A^T) = \{ \mathbb{x} \in \mathbb{R}^4 \mid A^T\mathbb{x} = \mathbb{0} \} = span(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix})$

- 求 $C(A)$

(a) $C(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \middle| \begin{matrix} -b_1 + b_2 + b_3 = 0 \\ -b_1 + b_4 = 0 \end{matrix} \right\}$ $\quad \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 1 & 1 & b_2 \\ 0 & 1 & b_3 \\ 1 & 2 & b_4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_1 \\ 0 & 0 & -b_1 + b_2 + b_3 \\ 0 & 0 & -b_1 + b_4 \end{array}\right]$

(b) $= span(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix})$ $\quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_4 \\ -b_3 + b_4 \\ b_3 \\ b_4 \end{bmatrix} = b_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

- $\begin{cases} \dim R(A) = \dim C(A) = 2 = r \\ \dim N(A) = 0 = n - r \\ \dim N(A^T) = 2 = m - r \end{cases}$

$Ax = b$ 的各種觀點

(甲)(聯立方程)
$$\begin{cases} x+y+2z=4 & : p_1 \\ y+z=3 & : p_2 \\ x+y+z=1 & : p_3 \\ 2x+y+2z=2 & : p_4 \end{cases}$$

(平面物件)
$$x = \begin{bmatrix}-2\\0\\3\end{bmatrix} \in p_1\cap p_2\cap p_3\cap p_4 = V = x + N(A) = x + \{0\} = \{x\}$$
法向量 A_1, A_2, A_3, A_4

(乙)(矩陣方程)
$$\begin{bmatrix}1&1&2\\0&1&1\\1&1&1\\2&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\3\\1\\2\end{bmatrix}$$
$$A \quad x \quad = \quad b$$

有解 $x = \begin{bmatrix}-2\\0\\3\end{bmatrix}$

(丙)(線性變換)
$$\begin{cases} T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \\ T_A(x) = b \end{cases}$$

$\exists\, x = \begin{bmatrix}-2\\0\\3\end{bmatrix} \in \mathbb{R}^3,\ T(x) = b$ (即 $b \in \underline{\underline{Im(T)}}$)

$$T: \begin{bmatrix}-2\\0\\3\end{bmatrix}_{\mathcal{E}} \longrightarrow \begin{bmatrix}4\\3\\1\\2\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}-2\\0\\3\end{bmatrix}_{\{a_1,a_2,a_3\}}$$

(丁)(線性組合)
$$x\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + y\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + z\begin{bmatrix}2\\1\\1\\2\end{bmatrix} = \begin{bmatrix}4\\3\\1\\2\end{bmatrix}$$
$$x a_1 + y a_2 + z a_3 = b$$

$$b = (-2)a_1 + 0a_2 + 3a_3 \in Span(a_1,a_2,a_3) = \underline{\underline{C(A)}}$$
坐標系 $\{a_1,a_2,a_3\}$ 下, 坐標 $= (-2,0,3)$

- $$\left[\begin{array}{ccc|cc}1&1&2&4&a\\0&1&1&3&b\\1&1&1&1&c\\2&1&2&2&d\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|cc}1&0&0&-2&-b+c\\0&1&0&0&-a+b+c\\0&0&1&3&a-c\\0&0&0&0&\underline{-a+b-c+d}\end{array}\right]$$

$b = \begin{bmatrix}4\\3\\1\\2\end{bmatrix}$, $x = \begin{bmatrix}-2\\0\\3\end{bmatrix}$

$b = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$, $x = \begin{bmatrix}-b+c\\-a+b+c\\a-c\end{bmatrix}$

(當 $-a+b-c+d=0$)

$$= Span(e_1,e_2,e_3)$$
$$\begin{cases} R(A) = \mathbb{R}^3, \\ N(A) = \{0\}, \end{cases}$$

$$C(A) = span(a_1,a_2,a_3) = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} \middle| -a+b-c+d=0\right\}$$
$$N(A^T) = Span\left(\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right)$$

$$\begin{cases} \dim R(A) = \dim C(A) = 3 \\ \dim N(A) = 0,\ \dim N(A^T) = 1 \end{cases}$$