$\mathbb{R}^n$ 子空間 V 的 基底(basis) 與 維度(dimension)

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($A\mathbf{x}=\mathbf{v}$ 有唯一解)

- **定義1**　$\mathcal{B}=\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ 是 V 的基底：$\forall \mathbf{v}\in V$, $\mathbf{v}=c_1\mathbf{v}_1+\dots+c_k\mathbf{v}_k$ uniquely (唯一)

(1) $\Leftrightarrow \begin{cases} \text{(i)}\ V=\text{span}(\mathbf{v}_1,\dots,\mathbf{v}_k) & : \mathbf{v}=c_1\mathbf{v}_1+\dots+c_k\mathbf{v}_k \\ \text{(ii)}\ \{\mathbf{v}_1,\dots,\mathbf{v}_k\}\ \text{線性獨立} & : c_1,\dots,c_k\ \text{唯一} \end{cases}$　($\mathcal{B}$ 是 V 之生成集) generate/span　($\mathcal{B}$ 是獨立集)

(2) V 的維度 $\dim(V)=k$

- **定義2**　$\mathcal{B}=\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ $\begin{cases} \text{線性獨立}: & c_1\mathbf{v}_1+\dots+c_k\mathbf{v}_k=\mathbf{0} \Rightarrow \begin{bmatrix}c_1\\ \vdots\\ c_k\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} \\ \text{線性相依}: & \exists \begin{bmatrix}c_1\\ \vdots\\ c_k\end{bmatrix}\neq\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix},\ c_1\mathbf{v}_1+\dots+c_k\mathbf{v}_k=\mathbf{0} \end{cases}$

(c_i 全部為0)　($A\mathbf{x}=\mathbf{0}$ 只有解 $=\mathbf{0}$)

(不全為0)　(〃 有 解 $\neq\mathbf{0}$)

(linear independent/dependent)

- **討論**　$\mathbb{R}^3$ 基底：$\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$, $\mathbf{v}_i\neq\mathbf{0}$

(1) (×) $\{\mathbf{v}_1\}$: $\text{Span}(\mathbf{v}_1)\neq\mathbb{R}^3$

(2) (×) $\{\mathbf{v}_1,\mathbf{v}_2\}$: $\text{span}(\mathbf{v}_1,\mathbf{v}_2)\neq\mathbb{R}^3$

(3) (×) $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ 共平面(相依)：$\mathbf{v}_3=c_1\mathbf{v}_1+c_2\mathbf{v}_2 \Rightarrow c_1\mathbf{v}_1+c_2\mathbf{v}_2-\mathbf{v}_3=0$, $\begin{bmatrix}c_1\\ c_2\\ -1\end{bmatrix}\neq\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

(○) $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ 不共平面(獨立)：

$\begin{cases} \text{(i)}\ \forall \mathbf{v}\in\mathbb{R}^3,\ \mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 & \text{(生成)} \\ \text{(ii)}\ c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0} \Longrightarrow \begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} & \text{(獨立)} \end{cases}$

(否則, 設 $c_2\neq0$, $\mathbf{v}_2=\frac{-1}{c_2}(c_1\mathbf{v}_1+c_3\mathbf{v}_3)$ (共平面, ×))

$\Downarrow$

(iii) $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$
$=d_1\mathbf{v}_1+d_2\mathbf{v}_2+d_3\mathbf{v}_3$

$\Rightarrow (c_1-d_1)\mathbf{v}_1+(c_2-d_2)\mathbf{v}_2+(c_3-d_3)\mathbf{v}_3=\mathbf{0}$

$\Rightarrow (c_1-d_1)=(c_2-d_2)=(c_3-d_3)=0$

$\Rightarrow c_1=d_1,\ c_2=d_2,\ c_3=d_3$　(唯一)

(4) (×) $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$

(i) $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ 共平面：$\mathbf{v}_3=\begin{cases}c_1\mathbf{v}_1+c_2\mathbf{v}_2\\ \mathbf{v}_3\end{cases}$ (不唯一), $c_1\mathbf{v}_1+c_2\mathbf{v}_2-\mathbf{v}_3+0\mathbf{v}_4=\mathbf{0}$

(ii) $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ 不共平面：$\mathbf{v}_4=\begin{cases}c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3\\ \mathbf{v}_4\end{cases}$ (不唯一), $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3-\mathbf{v}_4=\mathbf{0}$

(或) $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix}_{3\times4}\mathbf{x}=\mathbf{0}$ 有非0解

(有自由參數)

$\mathbb{R}^3$

(1) $\{v_1, v_2, v_3\}$：$\mathbb{R}^3$之基底 $\Longleftrightarrow \begin{cases} \text{(i) } \operatorname{Span}(v_1, v_2, v_3) = \mathbb{R}^3 \\ \text{(ii) } \{v_1, v_2, v_3\} \text{ 線性獨立} \end{cases} \Longleftrightarrow \forall\, b \in \mathbb{R}^3,\ b = c_1v_1 + c_2v_2 + c_3v_3$ (唯一)

($Ax = b$ 恰有一解)

(2) $\{v_1, v_2, v_3\}$ $\begin{cases} \text{線性獨立：} c_1v_1 + c_2v_2 + c_3v_3 = 0 \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & (Ax = 0 \text{ 只有0解}) \\ \text{〃 相依：} \exists \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ c_1v_1 + c_2v_2 + c_3v_3 = 0 & (\text{〃, 有非0解}) \end{cases}$

• **Example 1** $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ 為 $\mathbb{R}^3$ 之基底　　$\dim \mathbb{R}^3 = 3$

證 (i) (生成) $\forall\, v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3,\ v = ae_1 + be_2 + ce_3,\ \Rightarrow \operatorname{Span}(e_1, e_2, e_3) = \mathbb{R}^3$

(2) (獨立) $c_1e_1 + c_2e_2 + c_3e_3 = 0 \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \{e_1, e_2, e_3\}$ 線性獨立

• **Example 2** $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\{v_1, v_2, v_3\}$ 為 $\mathbb{R}^3$ 之基底

證 $\left[\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & a \\ 2 & 1 & 0 & 0 & b \\ 1 & 2 & 2 & 0 & c \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 2a - c \\ 0 & 1 & 0 & 0 & -4a + b + 2c \\ 0 & 0 & 1 & 0 & 3a - b - c \end{array}\right]$

(1) (獨立) $c_1v_1 + c_2v_2 + c_3v_3 = 0 \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(2) (生成) $\forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3,\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (2a - c)v_1 + (-4a + b + 2c)v_2 + (3a - b - c)v_3$

$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = v_1 - v_2 + 2v_3$

• **Remarks**

(1) dependent (相依)：$u = 2v + 3w \Rightarrow \begin{cases} v = \frac{1}{2}u - \frac{3}{2}w \\ w = \frac{1}{3}u - \frac{2}{3}v \end{cases} \Rightarrow u - 2v + 3w = 0$

(2) $\{1, x, x^2, \dots\}$ 為 $P = \{p(x) \mid p(x)\text{為多項式}\}$ 之基底, $p(x) = a_0 + a_1x + a_2x^2 + \cdots$

(3) $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}$ 為 $\mathcal{F} = \{f(x) \mid f(x)\text{: 連續函數}\}$ 之基底

(Fourier expansion) $f(x) = a_0 + a_1\sin x + a_2\sin 2x + a_3\sin 3x + \cdots$
$+ b_1\cos x + b_2\cos 2x + b_3\cos 3x + \cdots$

(4) Taylor expansion

(低頻)　　(高頻) noise

Linear dependence & Basis

$$c_1 v_1 + \cdots + c_k v_k = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = b$$
$$= A \quad x$$

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$V \lhd \mathbb{R}^n$, $\{v_1, \cdots, v_k\} \subset V$

定義:
(1) $\{v_1, \cdots, v_k\}$ 線性相依: $\exists \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, $c_1 v_1 + \cdots + c_k v_k = 0$, ($Ax = 0$, 有非 0 解)

(2) 〃 〃 獨立: $c_1 v_1 + \cdots + c_k v_k = 0 \Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ ($Ax = 0$, 只有 0 〃)

(3) 〃 V 的基底: (i) $\mathrm{Span}(v_1, \cdots, v_k) = V$
(ii) $\{v_1, \cdots, v_k\}$ 線性獨立, ($\dim V = k$)
($Ax = b$, 有唯一解)

Proposition 3.1 $V = \mathrm{Span}(v_1, \cdots, v_k)$, $v \in V$.

(1) $\{v_1, \cdots, v_k\}$ 獨立 $\Leftrightarrow$ $v = c_1 v_1 + \cdots + c_k v_k$ (唯一)

(2) 〃 相依 $\Leftrightarrow$ ① $\exists \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \neq \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix}$, $c_1 v_1 + \cdots + c_k v_k = d_1 v_1 + \cdots + d_k v_k$ (不唯一)

$\Leftrightarrow$ ② $\exists i$, $v_i = c_1 v_1 + \cdots + \widehat{c_i v_i} + \cdots + c_k v_k$ ($\wedge$: omit)

(3) 〃 基底 $\Leftrightarrow$ $v = c_1 v_1 + \cdots + c_k v_k$ (唯一), $(c_1, \cdots, c_k)$: v 关於 $\{v_1, \cdots, v_k\}$ 的座標

證明 (1) "$\Leftarrow$" $0 = c_1 v_1 + \cdots + c_k v_k = 0 v_1 + \cdots + 0 v_k \Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

"$\Rightarrow$" $\left.\begin{aligned} v &= c_1 v_1 + \cdots + c_k v_k \\ &= d_1 v_1 + \cdots + d_k v_k \end{aligned}\right\} \Rightarrow (c_1 - d_1) v_1 + \cdots + (c_k - d_k) v_k = 0 \Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix}$

(2) ① "$\Rightarrow$" $\left\{\begin{aligned} 0 &= c_1 v_1 + \cdots + c_k v_k \\ v &= d_1 v_1 + \cdots + d_k v_k \end{aligned}\right. \Rightarrow v = (c_1 + d_1) v_1 + \cdots + (c_k + d_k) v_k$, $\begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} \neq \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_k + d_k \end{bmatrix}$

"$\Leftarrow$" $\left\{\begin{aligned} v &= c_1 v_1 + \cdots + c_k v_k \\ &= d_1 v_1 + \cdots + d_k v_k \end{aligned}\right.$, $\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \neq \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} \Rightarrow (c_1 - d_1) v_1 + \cdots + (c_k - d_k) v_k = 0$
且 $\begin{bmatrix} c_1 - d_1 \\ \vdots \\ c_k - d_k \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

② "$\Rightarrow$" $c_1 v_1 + \cdots + c_i v_i + \cdots + c_k v_k = 0$
$(c_i \neq 0)$ $\Rightarrow v_i = -\frac{1}{c_i} (c_1 v_1 + \cdots + \widehat{c_i v_i} + \cdots + c_k v_k)$

"$\Leftarrow$" $v_i = c_1 v_1 + \cdots + \widehat{c_i v_i} + \cdots + c_k v_k \Rightarrow c_1 v_1 + \cdots + (-1) v_i + \cdots + c_k v_k = 0$

Proposition $U = \{v_1, v_2, v_3\} \subset V = \{v_1, v_2, v_3, v_4\}$ 則
(1) U 相依 $\Rightarrow$ V 相依
(2) V 獨立 $\Rightarrow$ U 獨立

證: (1) $c_1 v_1 + c_2 v_2 + c_3 v_3 + 0 \cdot v_4 = 0$, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

尋找基底？

• Example 2 (方程式) $V=\{\mathbb{x}\in\mathbb{R}^3 \mid x_1-x_2+2x_3=0\} \triangleleft \mathbb{R}^3$ 之基底？

解：

(i) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_2-2x_3\\x_2\\x_3\end{bmatrix}=x_2\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\mathbb{v}_1}+x_3\underbrace{\begin{bmatrix}-2\\0\\1\end{bmatrix}}_{\mathbb{v}_2}$

(ii) $c_1\mathbb{v}_1+c_2\mathbb{v}_2=\mathbb{0} \Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\left.\begin{array}{l}\text{Span}(\mathbb{v}_1,\mathbb{v}_2)=V\\ \{\mathbb{v}_1,\mathbb{v}_2\}\text{ 獨立}\end{array}\right\} \Rightarrow$ 基底, $\dim V=2$

註：高斯消去法所得參數式向量必為基底

• Example 3 (參數式) $\mathbb{v}_1=\begin{bmatrix}1\\1\\2\end{bmatrix}$, $\mathbb{v}_2=\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\mathbb{v}_3=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $V=\text{span}(\mathbb{v}_1,\mathbb{v}_2,\mathbb{v}_3) \triangleleft \mathbb{R}^3$ 之基底

解：

(i) $\left[\begin{array}{ccc|c}1&1&1&0\\1&-1&0&0\\2&0&1&0\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1&1&1&0\\0&2&1&0\\0&0&0&0\end{array}\right] \Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=c_2\begin{bmatrix}1\\1\\-2\end{bmatrix}$ $(\in N(A))$

$\Rightarrow \mathbb{v}_1+\mathbb{v}_2-2\mathbb{v}_3=\mathbb{0} \Rightarrow \begin{cases}(1)\ \{\mathbb{v}_1,\mathbb{v}_2,\mathbb{v}_3\}\text{ 相依}\\ (2)\ \mathbb{v}_3\in \text{Span}(\mathbb{v}_1,\mathbb{v}_2) \Rightarrow \text{span}(\mathbb{v}_1,\mathbb{v}_2)=V\end{cases}$ $\{\mathbb{v}_1,\mathbb{v}_2\}$ 獨立

(ii) $\mathbb{u}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbb{w}=\begin{bmatrix}2\\3\\5\end{bmatrix}\in V$? $\left[\begin{array}{cc|cc}1&1&1&2\\1&-1&1&3\\2&0&0&5\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc}1&1&1&2\\0&2&0&-1\\0&0&-2&0\end{array}\right] \Rightarrow \begin{cases}\mathbb{u}\notin V\\ \mathbb{w}=\frac{5}{2}\mathbb{v}_1-\frac{1}{2}\mathbb{v}_2\in V\end{cases}$

$c_1\mathbb{v}_1+c_2\mathbb{v}_2=\mathbb{u},\mathbb{w}$? $c_1=\frac{5}{2}$, $c_2=\frac{-1}{2}$ (但) $=2\mathbb{v}_1-\mathbb{v}_2+\mathbb{v}_3$

• Example 4 (Ex. 3.3-9) $\{\mathbb{u},\mathbb{v},\mathbb{w}\}$ 獨立 $\Rightarrow \{\mathbb{u}+\mathbb{v},\mathbb{v}+\mathbb{w},\mathbb{w}+\mathbb{u}\}$ 獨立

證：$c_1(\mathbb{u}+\mathbb{v})+c_2(\mathbb{v}+\mathbb{w})+c_3(\mathbb{w}+\mathbb{u})=\mathbb{0}$

$(c_1+c_3)\mathbb{u}+(c_1+c_2)\mathbb{v}+(c_2+c_3)\mathbb{w}=\mathbb{0}$

$\begin{cases}c_1+c_3=0\\c_1+c_2=0\\c_2+c_3=0\end{cases} \Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

• Example 5 $\begin{cases}\{\mathbb{v}_1,\dots,\mathbb{v}_n\}\ \mathbb{R}^n\text{ 之基底}\\ A_{n\times n}:\text{ nonsingular}\end{cases} \Rightarrow \{A\mathbb{v}_1,\dots,A\mathbb{v}_n\}\ \mathbb{R}^n$ 之基底

證：$\begin{cases}\forall\, \mathbb{b}\in\mathbb{R}^n\\ \exists 1\ \mathbb{x}\in\mathbb{R}^n\end{cases}$ $\mathbb{b}=A\mathbb{x}$ (∵ A nonsingular)

$=A(c_1\mathbb{v}_1+\cdots+c_n\mathbb{v}_n)$, $\exists 1 \begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}$ (∵ $\{\mathbb{v}_1,\dots,\mathbb{v}_n\}$ $\mathbb{R}^n$ 基底)

$=c_1(A\mathbb{v}_1)+\cdots+c_n(A\mathbb{v}_n)$

試證: $w_1=\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}, w_2=\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix}, \cdots, w_4=\begin{bmatrix}a_{14}\\a_{24}\\a_{34}\end{bmatrix}\in\mathbb{R}^3$

則 $\{w_1,w_2,w_3,w_4\}$ 相依

$V \lhd \mathbb{R}^n,\ \dim V = k$

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- **Proposition 4.1.** $\{v_1,\dots,v_k\}$: V之基底, $\{w_1,\dots,w_m\}\subset V,\ m>k$ $\Rightarrow \{w_1,\dots,w_m\}$ 相依 (獨立 $\Rightarrow m\le k$)

$(k=3,\ m=4)$

證: $0 = c_1w_1+c_2w_2+c_3w_3+c_4w_4$

$$= \begin{bmatrix} w_1 & w_2 & w_3 & w_4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} \quad \left(\text{存在 } \begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}\ne\begin{bmatrix}0\\0\\0\\0\end{bmatrix}?\right)$$

$$= \begin{bmatrix} v_1 & v_2 & v_3\end{bmatrix}\underline{\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}}$$

$$\begin{cases} w_1 = a_{11}v_1+a_{21}v_2+a_{31}v_3\\ w_2 = a_{12}v_1+a_{22}v_2+a_{32}v_3\\ w_3 = a_{13}v_1+a_{23}v_2+a_{33}v_3\\ w_4 = a_{14}v_1+a_{24}v_2+a_{34}v_3\end{cases}$$

$$\xRightarrow{\{v_i\}\text{獨立}} \underline{\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}} = \begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \xRightarrow{\text{自由參數}} \begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} \text{有非0解}$$

- **Proposition 3.2.** $\{v_1,\dots,v_k\}$ 獨立, 則 $\begin{cases}(1)\ u\in \mathrm{Span}(v_1,\dots,v_k) \Leftrightarrow \{v_1,\dots,v_k,u\} \text{ 相依}\\ (2)\ u\notin \quad '' \quad \Leftrightarrow \{\quad '' \quad\} \text{ 獨立}\end{cases}$

證: (1) $\begin{cases}``\Rightarrow" \quad u=c_1v_1+\cdots+c_kv_k\\ ``\Leftarrow" \quad (\text{設不全為0之 } c_i)\ c_1v_1+\cdots+c_kv_k+c_{k+1}u=0 \Rightarrow c_{k+1}\ne 0,\ u=\frac{-1}{c_{k+1}}(c_1v_1+\cdots+c_kv_k)\end{cases}$

- **Proposition 3.3.** (1) $\{w_1,\dots,w_m\}$ 相依 $\Rightarrow \exists\, w_i,\ \mathrm{span}(w_1,\dots,\hat{w_i},\dots,w_m)=\mathrm{Span}(w_1,\dots,w_m)$

(2) $\{w_1,\dots,w_m\}$ 獨立 $\Rightarrow \mathrm{Span}(w_1)\subsetneq \mathrm{Span}(w_1,w_2)\subsetneq\cdots\subsetneq\mathrm{Span}(w_1,w_2,\dots,w_m)$

證: (1) (c_i 不全為0) $c_1w_1+\cdots+\underbrace{c_i}_{\text{設}\ne0}w_i+\cdots+c_mw_m=0 \Rightarrow w_i=\frac{-1}{c_i}(c_1w_1+\cdots+\widehat{c_iw_i}+\cdots+c_mw_m)$

- **Proposition 3.4.** $A_{n\times n}$ nonsingular $\Leftrightarrow \{a_1,\dots,a_n\}$ $\mathbb{R}^n$之基底 (證: $\Leftrightarrow Ax=b$ 有唯一解)

- **Proposition 3.5.** $V\ne\{0\},\ V\lhd\mathbb{R}^n$ 存在基底

證:
$$\begin{cases} v_1 & & V=\mathrm{Span}(v_1)\ ?\\ v_2\in V-\mathrm{Span}(v_1) & \Rightarrow \{v_1,v_2\} \text{ 獨立}, & V=\mathrm{Span}(v_1,v_2)\ ?\\ v_3\in V-\mathrm{Span}(v_1,v_2) & \Rightarrow \{v_1,v_2,v_3\} \text{ 獨立}, & V=\mathrm{Span}(v_1,v_2,v_3)\ ?\\ \vdots & \vdots & \vdots\\ v_n\in V-\mathrm{span}(v_1,v_2,\dots,v_{n-1}) & \Rightarrow \{v_1,\dots,v_{n-1},v_n\} \text{ 獨立}, & V=\mathrm{Span}(v_1,v_2,\dots,v_n)\ ?\end{cases}$$

$\exists\ 1\le k\le n,\ V=\mathrm{Span}(v_1,\dots,v_k)$! 否則 $\exists\, u\in V-\mathrm{span}(v_1,v_2,\dots,v_n) \Rightarrow \{v_1,\dots,v_n,u\}$ 獨立 (※)

- **定理4.2** $\{v_1,\dots,v_k\}$, $\{w_1,\dots,w_m\}$ 均為 V 之基底 $\Rightarrow m=k$ 　證: $\begin{cases} m\le k\\ k\le m\end{cases}$ (Prop 4.1)

(若 $W \neq V$)

- **Proposition 4.3** $\begin{cases} W, V \lhd \mathbb{R}^n \\ W \subset V,\ \dim W = \dim V \end{cases} \Rightarrow W = V$

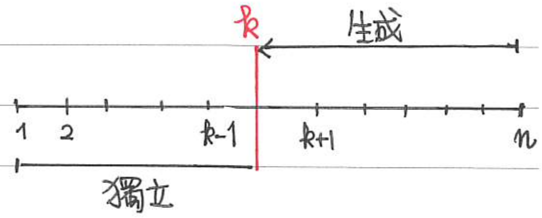
證: $\{w_1, \dots, w_k\}$ W 基底, $\exists u \in V - W$
則 $\{w_1, \dots, w_k, u\}$ 獨立, ※ (Prop4.1)

- **Proposition 4.4** $\begin{cases} \dim V = k \\ v_1, \dots, v_k \in V \end{cases}$ 則 $\mathrm{span}(v_1, \dots, v_k) = V \Leftrightarrow \{v_1, \dots, v_k\}$ 獨立 ($\Rightarrow V$ 之基底)

證: "$\Leftarrow$" $W = \mathrm{span}(v_1, \dots, v_k)$
"$\Rightarrow$" 若相依 $\xrightarrow{\text{Prop3.3}}$ ※

- **討論:** $\mathcal{B} = \{v_1, \dots, v_m\} \subset V$, $\dim V = k$

(1) $\mathcal{B}$: 基底 $\Leftrightarrow$ 獨立生成集
$\Leftrightarrow$ 最大 (k個) 獨立集
$\Leftrightarrow$ 最小 (k個) 生成集



(2) $\begin{cases} \text{(a) } \mathcal{B} \text{ 生成 } V \Rightarrow m \geq k \\ \text{(b) } \mathcal{B} \text{ 獨立 } \Rightarrow m \leq k \end{cases}$ (造基底) $\begin{matrix} m > k,\ m \searrow k \\ m < k,\ m \nearrow k \end{matrix}$

- **Example**

(a) $v_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\2\\4\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $v_4 = \begin{bmatrix}3\\4\\7\end{bmatrix}$, 則 $\{v_1, v_3\}$ 為 $V = \mathrm{span}(v_1, v_2, v_3, v_4)$ 基底

解: $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow V = C(A)$ 之基底: $\{v_1, v_3\}$

(或) $Ax = 0 \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}}_{u_1} + x_4 \underbrace{\begin{bmatrix}-3\\0\\-1\\1\end{bmatrix}}_{u_2}$, $\begin{cases} Au_1 = 0 \Rightarrow -2v_1 + v_2 = 0 \\ Au_2 = 0 \Rightarrow -3v_1 - v_3 + v_4 = 0 \end{cases}$

$\Rightarrow v_2, v_4 \in \mathrm{Span}(v_1, v_3)$

(b) $v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\3\\-1\end{bmatrix}$, 求 $\{v_1, v_2, v_3, v_4\}$ 為 $\mathbb{R}^4$ 之基底

解: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -1 \end{bmatrix}$, $Ax = 0 \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_3 \underbrace{\begin{bmatrix}2\\-3\\1\\0\end{bmatrix}}_{v_3} + x_4 \underbrace{\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}}_{v_4}$ (註: $v_3, v_4 \perp \{v_1, v_2\}$)

$N(A)$ 基底: $\{v_3, v_4\}$ $\Rightarrow \{v_1, v_2, v_3, v_4\}$: $\mathbb{R}^4$ 基底

驗算: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -1 \\ 2 & -3 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\dim = 4$

求四大子空間

$$A=\begin{bmatrix}1&1&0&1&4\\1&2&1&1&6\\0&1&1&1&3\\2&2&0&1&7\end{bmatrix}$$

求 $\begin{cases} R(A) = \mathrm{Span}(A_1, A_2, A_3, A_4) \text{ 之基底} \\ C(A) = \mathrm{Span}(a_1, a_2, \cdots, a_5) \\ N(A) = \{ x \in \mathbb{R}^5 \mid Ax = 0 \} \\ N(A^T) = \{ x \in \mathbb{R}^4 \mid A^T x = 0 \} \end{cases}$ $\begin{cases} m=4, \quad n=5 \\ r = \mathrm{rank}(A) = 3 \end{cases}$

解:

$$\begin{array}{r}\\ \\ \\ E_4 \rightarrow\end{array}\begin{bmatrix}1&0&-1&0\\-1&1&0&0\\1&-1&1&0\\-1&-1&1&1\end{bmatrix}\begin{bmatrix}1&1&0&1&4\\1&2&1&1&6\\0&1&1&1&3\\2&2&0&1&7\end{bmatrix}=\begin{bmatrix}1&0&-1&0&1\\0&1&1&0&2\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}\begin{array}{l}\leftarrow R_1\\ \leftarrow R_2\\ \leftarrow R_3\\ \\ \end{array}$$

$\uparrow a_1 \quad \uparrow a_2 \quad \uparrow a_4 \qquad (EA=R)$

子空間：基底

• $R(A)$: $\{R_1, R_2, R_3\} = \left\{ \begin{bmatrix}1\\0\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} \right\}$

$$\begin{cases} x_1 - x_3 + x_5 = 0 \\ x_2 + x_3 + 2x_5 = 0 \\ x_4 + x_5 = 0 \end{cases}$$

• $N(A)$: $\{u_1, u_2\} = \left\{ \begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\0\\-1\\1\end{bmatrix} \right\}$

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}x_3 - x_5\\-x_3-2x_5\\x_3\\-x_5\\x_5\end{bmatrix} = x_3 \underbrace{\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix}}_{u_1} + x_5 \underbrace{\begin{bmatrix}-1\\-2\\0\\-1\\1\end{bmatrix}}_{u_2}$$

• $C(A)$: $\{a_1, a_2, a_4\} = \left\{ \begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$

(獨立) $\begin{bmatrix}a_1 & a_2 & a_4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_4\end{bmatrix} = 0 \rightsquigarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_4\end{bmatrix} = 0 \Rightarrow \begin{cases}x_1=0\\x_2=0\\x_4=0\end{cases}$

(生成) $\begin{cases} Au_1 = 0 \\ Au_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 - a_2 + a_3 = 0 \\ -a_1 - 2a_2 - a_4 + a_5 = 0 \end{cases}$

$\Rightarrow a_3, a_5 \in \mathrm{span}(a_1, a_2, a_4)$

• $N(A^T)$: $\{E_4\} = \begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$

$E^T: \mathbb{R}^4 \to \mathbb{R}^4$ (1-1, onto)

$E^T y = x$ $(\forall x, \exists y)$

$0 = A^T x = A^T E^T y = R^T y \Leftrightarrow$

$$\begin{cases} R^T y = 0: \begin{bmatrix}1&0&0&0\\0&1&0&0\\-1&1&0&0\\0&0&1&0\\1&2&1&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} = y_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \\ x = E^T y: \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}1&-1&1&-1\\0&1&-1&-1\\-1&0&1&1\\0&0&0&1\end{bmatrix} y_4 \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = y_4\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} \end{cases}$$

$\uparrow E_4$

線性代數基本定理 Th4.6 Th4.10

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- **定理4.6** $\mathrm{rank}(A_{m\times n}) = r \Rightarrow \begin{cases} \dim R(A) = \dim C(A) = r \\ \dim N(A) = n-r \\ \dim N(A^T) = m-r \end{cases}$

- **推論4.7** (Nullity-rank th) $\begin{cases} \mathrm{null}(A) + \mathrm{rank}(A) = n \\ \dim \mathrm{Ker}(T_A) + \dim \mathrm{Im}(T_A) = n \end{cases}$ $\quad (\mathrm{null}(A) = \dim N(A))$

- **Proposition 4.8** $V \triangleleft \mathbb{R}^n,\ \dim V = k \Rightarrow \dim V^\perp = n-k$

  證：$\{v_1, \cdots, v_k\}$ V之基底，$A = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}_{k\times n} \Rightarrow \begin{cases} V = R(A) \\ V^\perp = N(A) \end{cases}$

- **定理4.9** $\left\{\begin{matrix} (1)\ V \perp V^\perp & (3)\ V \cap V^\perp = \{0\} \\ (2)\ (V^\perp)^\perp = V & (4)\ \mathbb{R}^n = V + V^\perp \end{matrix}\right\} \Rightarrow \mathbb{R}^n = V \oplus V^\perp$

  證：(2) $(V^\perp)^\perp \supset V,\ \dim (V^\perp)^\perp = n-(n-k) = k = \dim V$

  (4) $V + V^\perp \triangleleft \mathbb{R}^n \xRightarrow{(3)} \dim(V+V^\perp) = k + (n-k) = n \Rightarrow V+V^\perp = \mathbb{R}^n$

- **定理4.10** $\begin{cases} (1)\ R(A)^\perp = N(A), & N(A)^\perp = R(A), & \mathbb{R}^n = R(A) \oplus N(A) \\ (2)\ C(A)^\perp = N(A^T), & N(A^T)^\perp = C(A), & \mathbb{R}^m = C(A) \oplus N(A^T) \end{cases}$

- **定理4.11** $T' : R(A) \longrightarrow C(A),\ T'(x) = T_A(x),\ \forall x \in R(A)$

  $\begin{cases} (1)\ \mathrm{Ker}(T') = \{0\} & (T' : 1\text{-}1) \\ (2)\ \{v_1, \cdots, v_r\}\ R(A)\text{基底} \Rightarrow \{Av_1, \cdots, Av_r\}\ C(A)\text{基底} \\ (3)\ x = \underbrace{c_1 v_1 + \cdots + c_r v_r}_{\left[\begin{smallmatrix} c_1 \\ \vdots \\ c_r \end{smallmatrix}\right]_{\{v_i\}}} \in R(A) \Leftrightarrow T'(x) = \underbrace{c_1 Av_1 + \cdots + c_r Av_r}_{\left[\begin{smallmatrix} c_1 \\ \vdots \\ c_r \end{smallmatrix}\right]_{\{Av_i\}}} \in C(A) & (T' : \text{bijective}) \end{cases}$

  證：

  (1) $\mathrm{Ker}(T_A') = N(A) \cap R(A) = \{0\}$

  (2) $\{Av_1, \cdots, Av_r\}$ 獨立，$\because\ 0 = c_1 Av_1 + \cdots + c_r Av_r = A(c_1 v_1 + \cdots + c_r v_r)$

  $\Rightarrow c_1 v_1 + \cdots + c_r v_r = 0 \Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

- **Review** $\{x \mid Ax = b\} = x_0 + N(A) = \{x_0 + y \mid Ay = 0\}$
  $(Ax_0 = b)$

定理4.10, 定理, 4.11, Review

(1) $T: \mathbb{R}^n = R(A) \oplus N(A) \longrightarrow C(A) \triangleleft \mathbb{R}^m$ $\qquad \mathbb{R}^n/N(A) \cong R(A) \cong C(A)$

同構(isomorphic) 線性變換

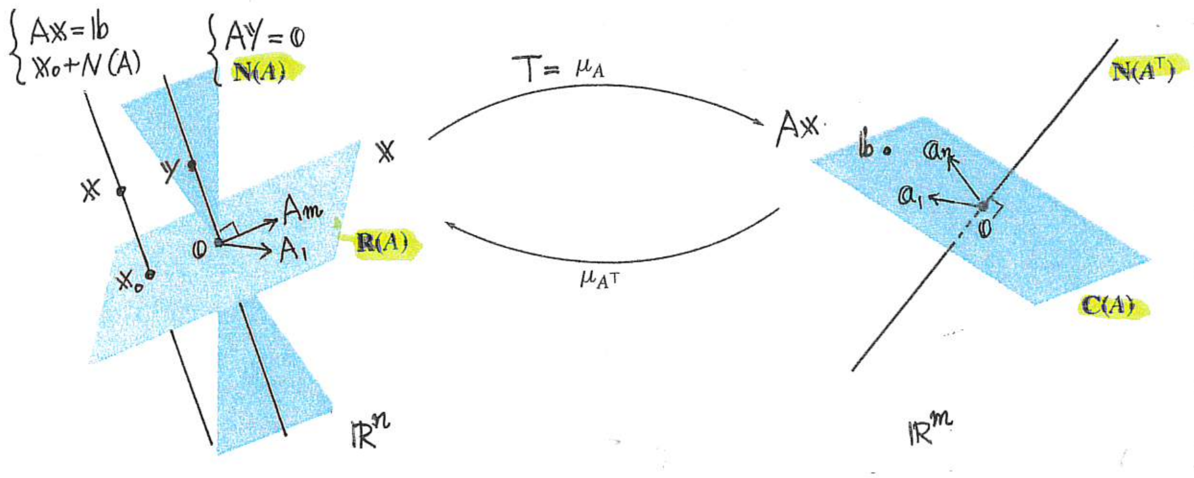
$$0 + y \longmapsto 0 + 0 = 0$$

$$x = x_0 + y \longmapsto b + 0 = b$$

$x_0 = c_1v_1+\cdots+c_rv_r = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}_{\{v_i\}}$ $\qquad b = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}_{\{Av_i\}} = c_1Av_1+\cdots+c_rAv_r$

$\dim R(A) = \dim C(A) = r$

(2) $\begin{cases} \dim N(A) = n-r \\ \dim N(A^T) = m-r \end{cases}$ $\qquad r$個獨立方程式(法向量 A_i) $\Rightarrow \dim N(A) = n-r$
(限制自由度)



定理12 $A_{n\times n}$: nonsingular $(\operatorname{rank}(A) = n)$, $\qquad T = \mu_A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$
(無自由參數)

$\Leftrightarrow A \overset{E}{\rightsquigarrow} I_n$

$\Leftrightarrow Ax = 0$ 只有0解 $\Leftrightarrow Ax = b$ 恰有一解: $Eb = A^{-1}b$ *

$\Leftrightarrow A^{-1}$ 存在 $(=E)$

$\Leftrightarrow N(A) = \operatorname{Ker}(T) = \{0\} \Leftrightarrow$ 1-1* $\Leftrightarrow \dim N(A) = 0$

$\Leftrightarrow R(A) = \mathbb{R}^n \Leftrightarrow \dim R(A) = n \Leftrightarrow \{A_i\}$ 基底 $\}$ 獨立 生成

$\Leftrightarrow C(A) = \operatorname{Im}(T) = \mathbb{R}^n \Leftrightarrow$ onto* $\Leftrightarrow \dim C(A) = n \Leftrightarrow \{a_j\}$ 〃 〃

$\Leftrightarrow \det(A) \neq 0$

$\Leftrightarrow$ 0不是A的 eigenvalue $\qquad$ (不存在 $x \neq 0$, $Ax = 0x$)

$\Leftrightarrow$ A有n個非0的 singular values

Nonsingular

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Invertible

m. The columns of A form a basis of $\mathbb{R}^n$.

$A\mathbf{x} = \mathbf{b}$.

n. $\text{Col } A = \mathbb{R}^n$

$R(A)$, $N(A)$

o. $\dim \text{Col } A = n$

$C(A)$,

p. $\text{rank } A = n$

q. $\text{Nul } A = \{\mathbf{0}\}$

r. $\dim \text{Nul } A = 0$

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

u. $(\text{Col } A)^{\perp} = \{\mathbf{0}\}$.

v. $(\text{Nul } A)^{\perp} = \mathbb{R}^n$.

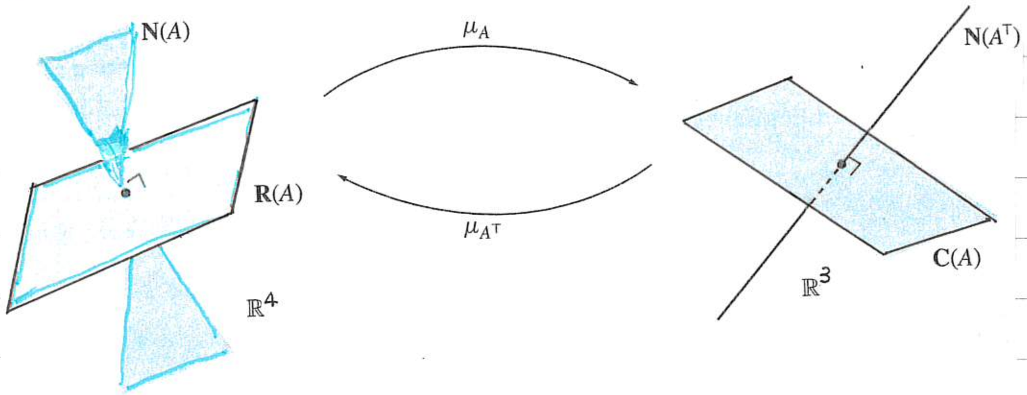
w. $\text{Row } A = \mathbb{R}^n$.

x. A has n nonzero singular values.

(甲)

$$A=\begin{bmatrix}1&-1&1&2\\1&0&1&1\\1&2&1&-1\end{bmatrix}\rightsquigarrow\begin{bmatrix}1&0&1&1\\0&1&0&-1\\0&0&0&0\end{bmatrix}$$

$$\begin{cases} R(A)=\text{span}\left(\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\right)\\ N(A)=\text{span}\left(\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right)\\ C(A)=\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}-1\\0\\2\end{bmatrix}\right)\\ N(A^T)=\text{Span}\left(\begin{bmatrix}2\\-3\\1\end{bmatrix}\right)\end{cases}$$



(乙)

$$\left[\begin{array}{ccc|c}1&1&1&0\\1&-1&0&1\\2&0&1&1\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&1&1&0\\0&2&1&-1\\0&0&0&0\end{array}\right]$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{2}-\frac{z}{2}\\\frac{-1}{2}-\frac{z}{2}\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{-1}{2}\\0\end{bmatrix}+\frac{-z}{2}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$= \mathbf{x}_0+\text{Span}\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)$$

$$= \{\mathbf{x}\mid A\mathbf{x}=\mathbf{b}\}$$

$$\begin{cases} R(A)=\text{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix}\right)=\{\mathbf{x}\mid x+y-2z=0\}\\ N(A)=\text{Span}\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)=\{\mathbf{x}\in\mathbb{R}^3\mid A\mathbf{x}=\mathbf{0}\}\\ C(A)=\text{Span}\left(\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\-1\\0\end{bmatrix}\right)\\ N(A^T)=\text{Span}\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)\end{cases}$$

$$=\text{Span}\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)$$