Abstract Vector Space

Abstract：研究共同數學性質，不談實例
(1) 抽象：抽取內在本質 (essence)
(2) 具象：表面現象 (details)

$V_{\mathbb{R}}$：vector space over $\mathbb{R}$ （實向量空間）

(1) $\begin{cases} \forall\, u, v \in V, & u+v \in V \text{ (vector 加法)} \\ \forall\, c \in \mathbb{R}, & cu \in V \text{ (scalar 乘法)} \end{cases}$ (2) $\begin{cases} \exists\, 0 \in V \\ \forall\, u \in V, & \exists\, -u \in V \text{ (反向量)} \end{cases}$

且 $\forall \begin{cases} u, v, w \in V \\ c, d \in \mathbb{R} \end{cases}$ 滿足

(1) $u + v = v + u$ (1)–(4) $(V, +)$ 加法交換群
(2) $(u+v)+w = u+(v+w)$
(3) $0 + u = u$ $\quad (\Rightarrow u + 0 = u)$
(4) $(-u) + u = 0$ $\quad (\Rightarrow u + (-u) = 0)$
(5) $1u = u$
(6) $c(du) = (cd)u$
(7) $c(u+v) = cu + cv$
(8) $(c+d)u = cu + du$

- **Property**

(1) $u+w = v+w \Rightarrow u = v$
(2) $0u = 0$
(3) $c0 = 0$
(4) $(-1)u = -u$

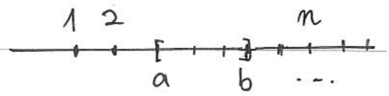
證
(1) $(u+w)+(-w) = (v+w)+(-w)$
(2) $0 + 0u = 0u = (0+0)u = 0u + 0u$
(3) $0 + c0 = c0 = c(0+0) = c0 + c0$
(4) $(-1)u + u = (-1+1)u = 0u = 0 = -u + u$
註：(4') $(-c)u = c(-u)$

- **Example**

(1) $\mathbb{R}^n = \{[x_1, \dots, x_n] \mid x_i \in \mathbb{R}\}$ $\begin{cases} u = [x_1, \dots, x_n] \\ v = [y_1, \dots, y_n] \end{cases}$ $\quad u+v = [x_1+y_1, \dots, x_n+y_n]$, $\quad cu = [cx_1, \dots, cx_n]$

(2) $\mathbb{R}^{\omega} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{R}\}$, $\begin{cases} u = (x_1, x_2, \dots) \\ v = (y_1, y_2, \dots) \end{cases}$ $\quad u+v = (x_1+y_1, x_2+y_2, \dots)$, $\quad cu = (cx_1, cx_2, \dots)$

(3) $\mathcal{F}([a,b]) = \{f \mid f : [a,b] \to \mathbb{R}\}$ $\begin{cases} (f+g)(t) = f(t) + g(t) \\ (cf)(t) = c f(t) \end{cases}$



(4) $\mathcal{M}_{m\times n} = \{A \mid A : m\times n \text{ 矩陣}\}$

(5) $L(V, W) = \{T \mid \text{線性變換 } T : V \to W\}$

- **定義** $(V, \langle , \rangle)$ Inner product space (內積空間)

V: 向量空間 且 $\forall u, v \in V$, $\langle u, v\rangle = u \cdot v \in \mathbb{R}$, 滿足

(1) $\langle u, v\rangle = \langle v, u\rangle$

(2) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ $\Rightarrow$ (2') $\langle u, v+w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(3) $\langle cu, w\rangle = c\langle u, w\rangle$, $\forall c \in \mathbb{R}$ (3') $\langle u, cw\rangle = c\langle u, w\rangle$

(4) $\begin{cases} \langle u, u\rangle \geq 0 \\ \langle u, u\rangle = 0 \Leftrightarrow u = 0 \end{cases}$

- **Example** (1) $\mathbb{R}^n$, $\begin{cases} u = [x_1, \cdots, x_n] \\ v = [y_1, \cdots, y_n] \end{cases}$ $\langle u, v\rangle = \sum_{i=1}^{n} x_i y_i$

(2) $\mathbb{R}^w$, $\begin{cases} u = (x_1, x_2, \cdots) \\ v = (y_1, y_2, \cdots) \end{cases}$ $\langle u, v\rangle = \sum_{i=1}^{\infty} x_i y_i$

(3) $C^0([a,b]) = \{ f \mid \text{continuous } f: [a,b] \to \mathbb{R} \}$, $\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$

(4) $M_{n\times n}$, $\langle A, B\rangle = tr(A^T B)$

- **定義** $\begin{cases} V_{\mathbb{R}}, W_{\mathbb{R}}: \text{vector space} \\ W \subset V \end{cases}$, $W_{\mathbb{R}}$ is subspace of $V_{\mathbb{R}}$ $(W \triangleleft V)$
(子空間)

- **定理** $W \triangleleft V \Leftrightarrow \begin{cases} (1)\ 0 \in W \\ (2)\ u, v \in W \Rightarrow u+v \in W \\ (3)\ c \in \mathbb{R} \Rightarrow cu \in W \end{cases}$

- **定義** (1) $\text{span}(v_1, \cdots, v_k) = \{ c_1 v_1 + \cdots + c_k v_k \mid c_i \in \mathbb{R} \} \triangleleft V$

(2) $\{v_1, \cdots, v_k\}$ linear independent: $c_1 v_1 + \cdots + c_k v_k = 0 \Rightarrow c_i = 0$, $\forall 1 \leq i \leq k$
(非 〃 dependent)

(3) $\{v_1, \cdots, v_k\}$: basis for W: $\begin{cases} \text{span}(v_1, \cdots, v_k) = W \\ \{v_1, \cdots, v_k\} \text{ indep.} \end{cases}$

(4) $\dim W = k$

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Example 1 $\mathcal{U} = \{$ upper triangular matrices $\}$
$\mathcal{L} = \{$ lower " " $\}$
$\mathcal{D} = \{$ diagonal " $\}$ $\lhd \mathcal{M}_{n\times n}$
$\mathcal{C}_M = \{A \in \mathcal{M}_{n\times n} \mid AM = MA\}$

Example 2 $\mathcal{C}^0(I) = \{ f \in \mathcal{F}(I) \mid f:$ conti $\}$
$\mathcal{C}^k(I) = \{$ " $\mid f^{(k)}:$ " $\}$
$\mathcal{C}^\infty(I) = \{$ " $\mid f: \infty$ 次可微$\}$
$P = \{$ polynomials $\}$
$P_k = \{a_0 + a_1 t + \cdots + a_k t^k \mid a_i \in \mathbb{R}\}$

註: $P_0 \lhd P_1 \lhd P_2 \lhd \cdots \lhd P_k \lhd \cdots \lhd P \lhd \mathcal{C}^\infty(\mathbb{R}) \lhd \cdots \lhd \mathcal{C}^2(\mathbb{R}) \lhd \mathcal{C}^1(\mathbb{R}) \lhd \mathcal{C}^0(\mathbb{R})$
$\lhd \mathcal{F}(\mathbb{R})$

Example 3 P_2: 基底 (1) $\mathcal{B}_1 = \{1, t, t^2\}$ $\quad \dim P_2 = 3$
(2) $\mathcal{B}_2 = \{t+1, t^2+2, t^2-t\}$
証 $\mathcal{B}_2$: indep: $\because$ $a(t+1) + b(t^2+2) + c(t^2-t) = 0$
$(b+c)t^2 + (a-c)t + (a+2b) = 0$
$$\begin{cases} b + c = 0 \\ a - c \\ a + 2b = 0 \end{cases} \Rightarrow a = b = c = 0$$

Example 4: $V = \{ f \in \mathcal{C}^1(\mathbb{R}) \mid f'(t) = f(t), t \in \mathbb{R}\} \lhd \mathcal{C}^1(\mathbb{R})$, basis: $\{e^t\}$

証: (i) $0 \in V$
(ii) $f, g \in V \Rightarrow f+g \in V$ $\quad \because (f+g)'(t) = f'(t) + g'(t) = f(t) + g(t) = (f+g)(t)$
(iii) $f \in V \Rightarrow cf \in V$
(a) $e^t \in V$
(b) $f(t) \in V \Rightarrow f(t) = ce^t$ $\quad \because \left(f(t)e^{-t}\right)' = f'(t)e^{-t} - f(t)e^{-t} = 0$
$\Rightarrow f(t)e^{-t} = c$

Inner product space

定義: (1) $\|u\| = \sqrt{\langle u,u\rangle}$

(2) $d(u,v) = \|u-v\|$

(3) $\measuredangle(u,v) = \cos^{-1}\dfrac{\langle u,v\rangle}{\|u\|\,\|v\|}$ (定理(2))

(4) $u \perp v \iff \langle u,v\rangle = 0$

(5) $W^{\perp} = \{u \in V \mid u \perp W\}$

定理 (1) $\|cv\| = |c|\,\|v\|$

(2) $|\langle v,w\rangle| \le \|v\|\,\|w\|$ (Cauchy 不等式)

(3) $\|v+w\| \le \|v\| + \|w\|$ (三角 〃)

Example 1 (1) $\mathbb{R}^n$: $(a_1b_1+\cdots+a_nb_n)^2 \le (a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)$

(2) $C^0([a,b])$: $\left(\int_a^b f(t)g(t)\,dt\right)^2 \le \left(\int_a^b f^2(t)\,dt\right)\left(\int_a^b g^2(t)\,dt\right)$

Example 2 $C^0([0,1])$: (1) $\|x+1\| = \sqrt{\langle x+1, x+1\rangle} = \sqrt{\int_0^1 (x+1)^2\,dt} = \sqrt{\frac{7}{3}}$

(2) $d(x^2,x) = \|x^2-x\| = \sqrt{\int_0^1 (x^2-x)^2\,dx} = \sqrt{\frac{1}{30}}$

Example 3
(偶函數)
(奇 〃)

$$\left\{\begin{array}{l} V = C^0([-1,1]), \quad \langle f,g\rangle = \int_{-1}^1 f(t)g(t)\,dt \\ U = \{f \in V \mid f(-t) = f(t),\ \forall t \in [-1,1]\} \\ W = \{f \in V \mid f(-t) = -f(t),\ \quad〃 \quad\} \end{array}\right. \Rightarrow \left\{\begin{array}{l} (1)\ U, W \lhd V \\ (2)\ U \perp W \\ (3)\ U^{\perp} = W \\ (4)\ V = U \oplus W \end{array}\right.$$

證 (2) $\begin{cases} g \in U \\ h \in W \end{cases}$ $\langle g,h\rangle = \int_{-1}^1 g(t)h(t)\,dt \overset{t=-s}{=\!=\!=} -\int_{-1}^1 g(s)h(s)\,ds \Rightarrow \langle g,h\rangle = 0$

(4) $V = U + W$ $\because f(t) = \underbrace{\tfrac{1}{2}[f(t)+f(-t)]}_{(偶)} + \underbrace{\tfrac{1}{2}[f(t)-f(-t)]}_{(奇)} \xrightarrow[U\perp W]{} \underset{(唯一)}{V = U \oplus W}$

(3) "$\supset$"

"$\subset$": $\underset{U^{\perp}}{\underset{\cap}{f}} = \underset{U}{\underset{\cap}{g}} + \underset{W}{\underset{\cap}{h}} \Rightarrow 0 = \langle f,g\rangle = \langle g,g\rangle + \langle h,g\rangle = \langle g,g\rangle$

$\Rightarrow g = 0$