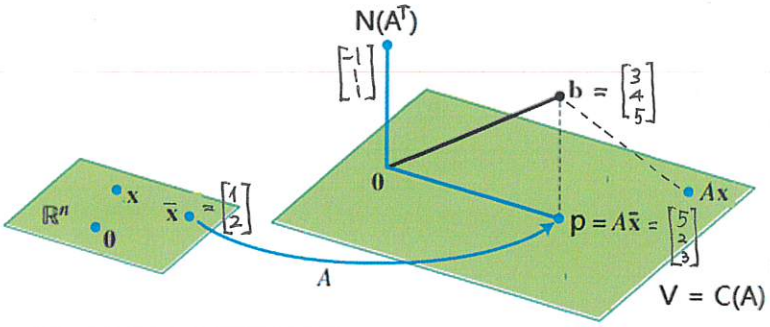
Chapter 4 Projections and Linear Transformations

Projection & Least square solution

- $\begin{cases} V = C(A) \triangleleft \mathbb{R}^m & \mathbb{R}^m = C(A) \oplus N(A^T) \\ \mathbf{b} \in \mathbb{R}^m & \mathbf{b} = \mathbf{p} + (\mathbf{b}-\mathbf{p}) \end{cases}$



- $A_{m\times n}\mathbf{x} = \mathbf{b}$

(a) $\mathbf{b} \in C(A)$ 有解, $\mathbf{b} = \begin{bmatrix}5\\2\\3\end{bmatrix}$, $\begin{bmatrix}1&2\\0&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5\\2\\3\end{bmatrix}$, $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

$A\bar{\mathbf{x}} = \mathbf{p}$ 有解

(b) $\mathbf{b} \notin C(A)$ 無解, $\mathbf{b} = \begin{bmatrix}3\\4\\5\end{bmatrix}$, $\begin{bmatrix}1&2\\0&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\4\\5\end{bmatrix}$, $\mathbf{p} = \mathrm{proj}_V \mathbf{b} = \begin{bmatrix}5\\2\\3\end{bmatrix}$, LS 解 $\bar{\mathbf{x}} = \begin{bmatrix}1\\2\end{bmatrix}$

- **定理** $\bar{\mathbf{x}}$: $A\mathbf{x} = \mathbf{b}$ 的 LS 解 $(A\bar{\mathbf{x}} = \mathbf{p})$

(1) $\|\mathbf{b} - A\bar{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|,\ \forall \mathbf{x} \in \mathbb{R}^n$ (最佳近似解)

(2) $A^TA\bar{\mathbf{x}} = A^T\mathbf{b}$ (Normal Eq.) $(\because \mathbf{b} - A\bar{\mathbf{x}} \in N(A^T) \Rightarrow A^T(\mathbf{b} - A\bar{\mathbf{x}}) = \mathbf{0})$

(3) $\bar{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$ $\mathbf{b} = \begin{bmatrix}3\\4\\5\end{bmatrix}, A = \begin{bmatrix}1&2\\0&1\\1&1\end{bmatrix} \Rightarrow \bar{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix}1\\2\end{bmatrix}$

(4) $\mathbf{p} = \mathrm{proj}_V \mathbf{b} = A(A^TA)^{-1}A^T\mathbf{b}$

(5) Projection matrix $P_V = A(A^TA)^{-1}A^T$ $(P_V^2 = P_V)\ (V = \mathrm{span}(\mathbf{v}_1, \cdots, \mathbf{v}_k),\ A = \begin{bmatrix}\mathbf{v}_1 \cdots \mathbf{v}_k\end{bmatrix})$

- **Lemma** $\mathrm{rank}(A) = n \Rightarrow A^TA$ invertible $(\because \underline{A^TA\mathbf{x} = \mathbf{0}} \Rightarrow A\mathbf{x}\cdot A\mathbf{x} = \mathbf{x}\cdot A^TA\mathbf{x} = \mathbf{0}$
$\Rightarrow A\mathbf{x} = \mathbf{0} \Rightarrow \underline{\mathbf{x} = \mathbf{0}}$

- **討論** $P_V = A(A^TA)^{-1}A^T$

(1) $\mathbf{b} \in C(A)$: $P_V\mathbf{b} = \mathbf{b}$ $\because \mathbf{b} = A\mathbf{x}$

(2) $\mathbf{b} \in N(A^T)$: $P_V\mathbf{b} = \mathbf{0}$ $A^T\mathbf{b} = \mathbf{0}$

(3) $V = \mathrm{Span}(\mathbf{a})$: $P_V\mathbf{b} = \frac{1}{\|\mathbf{a}\|^2}(\mathbf{a}\mathbf{a}^T)\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$ $A = \mathbf{a}$

(4) $V = \left\{\mathbf{x} \in \mathbb{R}^3 \,\middle|\, x - 2y + z = 0\right\} = \mathrm{Span}(\underset{\mathbf{u}}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\mathbf{v}}{\begin{bmatrix}-1\\0\\1\end{bmatrix}})$ $A = \begin{bmatrix}1&-1\\1&0\\1&1\end{bmatrix}$

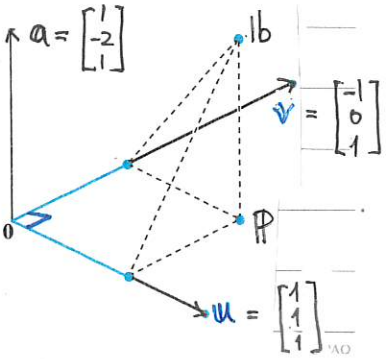
(甲) $P_V = A(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix}5&2&-1\\2&2&2\\-1&2&5\end{bmatrix}$

(乙) $P_V = I_3 - \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^T = \frac{1}{6}\left[\ 〃\ \right]$

(丙) $P_V = \begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}\left(\begin{bmatrix}\mathbf{u}^T\\\mathbf{v}^T\end{bmatrix}\begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{u}^T\\\mathbf{v}^T\end{bmatrix}$ $(\mathbf{u} \perp \mathbf{v})$

$= \frac{\mathbf{u}\mathbf{u}^T}{\|\mathbf{u}\|^2} + \frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2} = \frac{1}{6}\left[\ 〃\ \right]$

$P_V\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} + \frac{\mathbf{b}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$ (定理3)

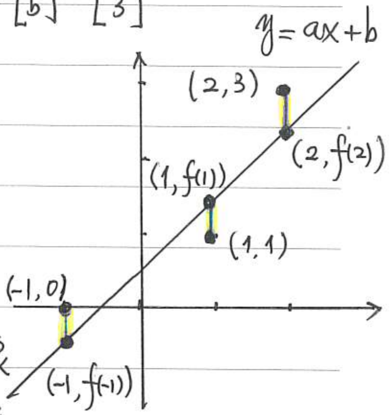


Data fitting (Least square, Regression)

Example 1 求 $y = f(x) = ax+b$ thru $(-1,1), (1,5), (2,7)$

解
$$\begin{cases} f(-1) = -a+b = 1 \\ f(1) = \ \ a+b = 5 \\ f(2) = 2a+b = 7 \end{cases} \Rightarrow \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{array}{c} \begin{matrix} x & 1 \end{matrix} \\ \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$A\mathbf{x} = \mathbf{b}$$

Example 2 求 $y = f(x) = ax+b \begin{Bmatrix} \text{thru} \\ \text{best fits} \end{Bmatrix} \begin{cases} (-1,0) \\ (1,1) \\ (2,3) \end{cases}$



最小平方(迴歸)直線
LS (regression) line

解(1)
$$\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{array}{c} \begin{matrix} x & 1 \end{matrix} \\ \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad \text{無解}$$
$$A \quad \mathbf{x} \quad = \quad \mathbf{b}$$

(2) 求 LS 解

$$\underset{\mathbf{x}=\begin{bmatrix} a \\ b \end{bmatrix}}{\text{Min}} \ D = [f(-1)-0]^2 + [f(1)-1]^2 + [f(2)-3]^2 = \left\| \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\|^2 = \|A\mathbf{x} - \mathbf{b}\|^2$$
$$= [(-a+b)-0]^2 + [(a+b)-1]^2 + [(2a+b)-3]^2$$

(甲) 微積分:

$$\begin{cases} \dfrac{\partial D}{\partial a} = 2[(-a+b-0)](-1) + 2[(a+b)-1](1) + 2[(2a+b)-3](2) = 0 \\ \dfrac{\partial D}{\partial b} = 2[\quad '' \quad](1) + 2[\quad '' \quad](1) + 2[\quad '' \quad](1) = 0 \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} -a+b \\ a+b \\ 2a+b \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(乙) 線性代數:
$$A^T (A\bar{\mathbf{x}} - \mathbf{b}) = \mathbf{0}$$

$$\therefore \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} = \bar{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{14}\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 13 \\ 10 \end{bmatrix}$$

$$y = \frac{13}{14}x + \frac{10}{14}$$

Example 3 求拋物線 $y = f(x) = ax^2 + bx + c$ best fits
$$\begin{cases} (1,-1) = (x_1, y_1) \\ (2,4) = (x_2, y_2) \\ (3,5) \qquad \vdots \\ (4,7) \\ (5,11) = (x_5, y_5) \end{cases}$$

解
$$\begin{cases} f(1) = a + b + c = -1 \\ f(2) = 4a + 2b + c = 4 \\ f(3) = 9a + 3b + c = 5 \\ f(4) = 16a + 4b + c = 7 \\ f(5) = 25a + 5b + c = 11 \end{cases} \Rightarrow \begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x^2 & x & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix} \end{array} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 5 \\ 7 \\ 11 \end{bmatrix}$$

$$A \, \mathbb{x} = \mathbb{b}$$

LS 解

$$\underset{\mathbb{x} = [a\ b\ c]}{\text{Min}}\ D = [f(1)+1]^2 + [f(2)-4]^2 + [f(3)-5]^2 + [f(4)-7]^2 + [f(5)-11]^2 = \left\| \begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 5 \\ 7 \\ 11 \end{bmatrix} \right\|^2 = \| A\mathbb{x} - \mathbb{b} \|^2$$

$$= \sum_{i=1}^{5} [f(x_i) - y_i]^2 = \sum_{i=1}^{5} [(ax_i^2 + bx_i + c) - y_i]^2$$

微積分

$$\begin{cases} \frac{\partial D}{\partial a} = \sum_{i=1}^{5} 2[f(x_i) - y_i]\, x_i^2 = 0 \\ \frac{\partial D}{\partial b} = \sum_{i} 2[\qquad]\, x_i = 0 \\ \frac{\partial D}{\partial c} = \sum_{i} 2[\qquad] \cdot 1 = 0 \end{cases} \Rightarrow \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ f(x_5) \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \right) = \mathbb{0}$$

線性代數

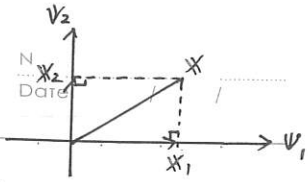
$$A^T (A\bar{\mathbb{x}} - \mathbb{b}) = \mathbb{0}$$

Example 4 $y = f(x) = a + bx + c\sin x + de^x$ best fits
$$\begin{cases} (1,-1) \\ (2,4) \\ (3,5) \\ (4,7) \\ (5,11) \end{cases}$$

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ f(x_5) \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1 & x & \sin x & e^x \end{array} \\ \begin{bmatrix} 1 & 1 & \sin 1 & e^1 \\ 1 & 2 & \sin 2 & e^2 \\ 1 & 3 & \sin 3 & e^3 \\ 1 & 4 & \sin 4 & e^4 \\ 1 & 5 & \sin 5 & e^5 \end{bmatrix} \end{array} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 5 \\ 7 \\ 11 \end{bmatrix}$$

$$\begin{cases} \text{thru: } A\mathbb{x} = \mathbb{b} \\ \text{best fits: } A^TA\bar{\mathbb{x}} = A^T\mathbb{b} \end{cases}$$

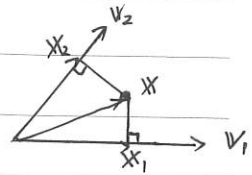
Orthogonal Bases 正交基底



定義 $\{v_1, \dots, v_k\}$ orthogonal : $v_i \cdot v_j = 0 \quad i \neq j$

$\{ \quad 〃 \quad \}$ orthonormal : $v_i \cdot v_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\begin{cases} x = x_1 + x_2 \\ x \neq x_1 + x_2 \end{cases}$



定理1' $\{u, v, w\}$ orthogonal $\Rightarrow$ independent

證 $au + bv + cw = 0 \overset{\cdot u}{\Longrightarrow} a = 0$

$\overset{\cdot v}{\Longrightarrow} b = 0$

$\cdot w \quad c = 0$

定理2' $\{u, v, w\}$ basis for V, 則 $\{u, v, w\}$ orthogonal $\Leftrightarrow \forall x \in V$, $x = \frac{x \cdot u}{\|u\|^2} u + \frac{x \cdot v}{\|v\|^2} v + \frac{x \cdot w}{\|w\|^2} w$

證 "$\Rightarrow$" $x = au + bv + cw$, $x \cdot u = a\|u\|^2$

"$\Leftarrow$" (取 $x = u$) $u = \frac{u \cdot u}{\|u\|^2} u + \frac{u \cdot v}{\|v\|^2} v + \frac{u \cdot w}{\|w\|^2} w \underset{(唯一性)}{\Longrightarrow} \begin{cases} u \cdot v = 0 \\ u \cdot w = 0 \end{cases}$

$= 1 \, u + 0 \, v + 0 \, w$

$(\dim V = 2)$

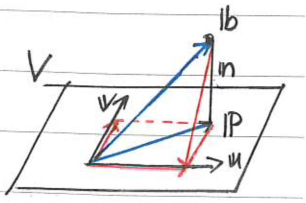
定理3' $V \triangleleft \mathbb{R}^3$, $\{u, v\}$ orthogonal basis $\Leftrightarrow \forall b \in \mathbb{R}^3$, $\text{proj}_V b = p = \frac{b \cdot u}{\|u\|^2} u + \frac{b \cdot v}{\|v\|^2} v$

證 "$\Rightarrow$" $p = \frac{p \cdot u}{\|u\|^2} u + \frac{p \cdot v}{\|v\|^2} v$,

$= \frac{b \cdot u}{\|u\|^2} u + \frac{b \cdot v}{\|v\|^2} v \qquad (\because b = p + n \Rightarrow b \cdot u = p \cdot u)$

或三垂線定理

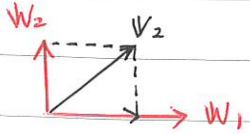
"$\Leftarrow$" (定理2') $\forall b \in V$



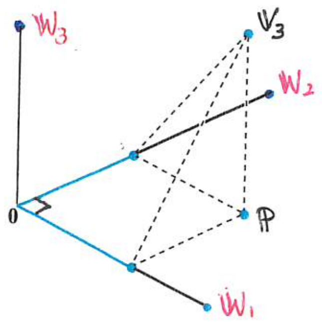
定理4' (Gram-Schmidt process) $\underset{\text{basis}}{\{v_1, v_2, v_3\}} \xrightarrow{正交化} \underset{\text{orthogonal basis}}{\{w_1, w_2, w_3\}} \xrightarrow{q_i = \frac{w_i}{\|w_i\|}} \underset{\text{orthonormal basis}}{\{q_1, q_2, q_3\}}$

Process

$$\begin{cases} (1) \ w_1 = v_1 \\ (2) \ w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1 \\ (3) \ w_3 = v_3 - \underbrace{\frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2}_{p} \end{cases}$$



$$\Rightarrow \begin{cases} (1) \ \{w_1, w_2, w_3\} \text{ orthogonal} \\ (2) \ \text{Span}(w_1, w_2, w_3) = \text{span}(v_1, v_2, v_3) \end{cases}$$



Example $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$

$$\begin{cases} w_1 = v_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1 = v_2 - w_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \\ w_3 = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2 = v_3 - 2w_1 + \frac{1}{2} w_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

$$\begin{cases} q_1 = \frac{1}{2} w_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ q_2 = \frac{1}{2\sqrt{2}} w_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \\ q_3 = \frac{1}{\sqrt{2}} w_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

Orthogonal Bases

定理 1 $\{v_1, \cdots, v_k\}$ orthogonal $\Rightarrow$ independent

定理 2 $\{v_1, \cdots, v_k\}$ basis for V, 則 $\{v_1, \cdots, v_k\}$ orthogonal $\Longleftrightarrow \forall x \in V,\ x = \sum_{i=1}^{k} \frac{x \cdot v_i}{\|v_i\|^2} v_i$

定理 3 $V \lhd \mathbb{R}^n$, $\dim V = k$, 則 $\{v_1, \cdots, v_k\}$ orthogonal basis $\Longleftrightarrow \forall b \in \mathbb{R}^n,\ \text{proj}_V b = \sum_{i=1}^{k} \frac{b \cdot v_i}{\|v_i\|^2} v_i$

定理 4 (Gram-Schmidt Process) basis $\{v_1, \cdots, v_k\} \xrightarrow{\text{正交化}} \underset{\text{orthogonal}}{\{w_1, \cdots, w_k\}} \xrightarrow{\text{單位長}} \underset{\text{orthonormal}}{\{q_1, \cdots, q_k\}}$

Process

$$\begin{cases} (1)\ w_1 = v_1 \\ (2)\ w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1 \\ (3)\ w_3 = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2 \\ \quad \vdots \\ (k)\ w_k = v_k - \frac{v_k \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_k \cdot w_2}{\|w_2\|^2} w_2 - \cdots - \frac{v_k \cdot w_{k-1}}{\|w_{k-1}\|^2} w_{k-1} \end{cases} \Rightarrow \begin{cases} (1)\ \{w_1, \cdots, w_k\} \text{ orthogonal} \\ (2)\ \text{Span}(w_1, \cdots, w_k) = \text{Span}(v_1, \cdots, v_k) \end{cases}$$

QR-decomposition

$$\begin{cases} v_1 = w_1 & = 2q_1 \\ v_2 = w_1 + w_2 & = 2q_1 + 2\sqrt{2}\,q_2 \\ v_3 = 2w_1 - \frac{1}{2}w_2 + w_3 & = 4q_1 - \sqrt{2}\,q_2 + \sqrt{2}\,q_3 \end{cases} \Rightarrow \underset{A}{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}_{4\times3}} \underset{=}{=} \underset{Q}{\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}_{4\times3}} \underset{R}{\begin{bmatrix} 2 & 2 & 4 \\ 0 & 2\sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}_{3\times3}}$$

- **定理** (QR-decomposition)

$$A_{m\times n} = Q_{m\times n} R_{n\times n} \quad \begin{cases} Q^TQ = I_n \text{ (columns orthonormal)} \\ R: \text{upper triangular, nonsingular, 对角線} > 0 \end{cases}$$

- **Remarks**

(1) LS 解: $\bar{x} = R^{-1}Q^Tb$ $\quad \because \bar{x} = (A^TA)^{-1}A^Tb = (R^TQ^TQR)^{-1}R^TQ^Tb = R^{-1}Q^Tb$

(2) Projection matrix $P_V = QQ^T$ $\quad \because P_V = A(A^TA^{-1})A^T = QRR^{-1}Q^T$

$$= q_1q_1^T + q_2q_2^T + q_3q_3^T \qquad P_V b = (b \cdot q_1)q_1 + (b \cdot q_2)q_2 + (b \cdot q_3)q_3$$

$$= P_1 + P_2 + P_3$$

Change of Basis

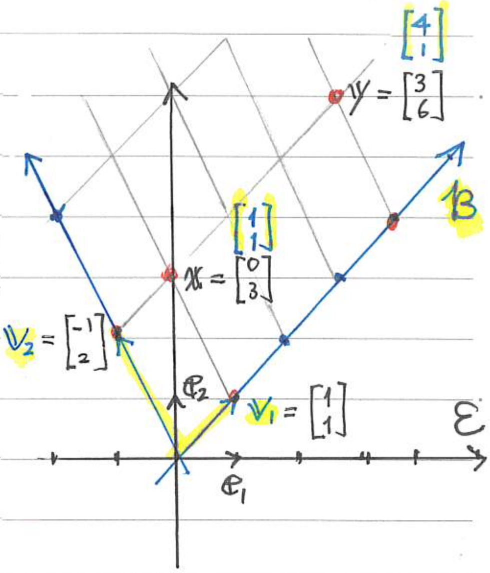
- **定理** $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathbb{R}^n$: 两基底 $\mathcal{E}=\{e_1,\dots,e_n\}$, $\mathcal{B}=\{v_1,\dots,v_n\}$

(1) 点X在 $\overset{\mathcal{E}}{\mathcal{B}}$ 座標 $\begin{aligned} x &= c_1v_1+\cdots+c_nv_n \\ x' &= C_{\mathcal{B}}(x)=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\end{aligned} \Rightarrow \begin{cases} x = P\,x' \\ x' = P^{-1}x \end{cases}$, $P=\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ $\mathcal{B}\to\mathcal{E}$ 座標變換矩陣

(2) $\begin{cases} [T]_{\mathcal{E}} = A = \begin{bmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{bmatrix} \\ [T]_{\mathcal{B}} = A' = \begin{bmatrix} | & & | \\ C_{\mathcal{B}}(T(v_1)) & \cdots & C_{\mathcal{B}}(T(v_n)) \\ | & & | \end{bmatrix} \end{cases} \Rightarrow \begin{cases} A = PA'P^{-1} \\ A' = P^{-1}AP \end{cases}$ $\because Ax=y \Rightarrow APx'=Py'$

$A'x'=y'$

- **Example** $T: \mathbb{R}^2 \to \mathbb{R}^2$



$$\begin{tikzcd} \mathcal{E}=\{e_1, e_2\} & x=\begin{bmatrix}0\\3\end{bmatrix} \arrow[r, "A=\begin{bmatrix}3&1\\2&2\end{bmatrix}"] & y=\begin{bmatrix}3\\6\end{bmatrix} \arrow[d, "P^{-1}=\frac{1}{3}\begin{bmatrix}2&1\\-1&1\end{bmatrix}"] \\ \mathcal{B}=\Big\{\underset{v_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}-1\\2\end{bmatrix}}\Big\} & x'=\begin{bmatrix}1\\1\end{bmatrix} \arrow[u, "\begin{bmatrix}1&-1\\1&2\end{bmatrix}=P"] \arrow[r, "A'=\begin{bmatrix}4&0\\0&1\end{bmatrix}"] & y'=\begin{bmatrix}4\\1\end{bmatrix} \end{tikzcd}$$

$$A' = P^{-1}\ A\ P$$

$$\begin{bmatrix}4&0\\0&1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2&1\\-1&1\end{bmatrix}\begin{bmatrix}3&1\\2&2\end{bmatrix}\begin{bmatrix}1&-1\\1&2\end{bmatrix}$$

$$y'=\begin{bmatrix}4\\1\end{bmatrix}\leftarrow\underset{y}{\begin{bmatrix}3\\6\end{bmatrix}}\leftarrow\underset{x}{\begin{bmatrix}0\\3\end{bmatrix}}\leftarrow\begin{bmatrix}1\\1\end{bmatrix}=x'$$ ($A\sim A'$, A is similar to A')

- **Remarks** $A=\begin{bmatrix}3&1\\2&2\end{bmatrix}_{\mathcal{E}}$ $A'=\begin{bmatrix}4&0\\0&1\end{bmatrix}_{\mathcal{B}}=P^{-1}AP$, $P=\begin{array}{c} \begin{matrix}v_1 & v_2\end{matrix} \\ \begin{bmatrix}1&-1\\1&2\end{bmatrix}\end{array}$ (A: 可对角化)

(1) $\begin{cases} T(v_1) = 4v_1, \\ T(v_2) = 1\,v_2, \end{cases}$ $\begin{matrix} \begin{bmatrix}1\\1\end{bmatrix}\to\begin{bmatrix}4\\4\end{bmatrix}, & \begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}4\\0\end{bmatrix}, \\ \begin{bmatrix}-1\\2\end{bmatrix}\to\begin{bmatrix}-1\\2\end{bmatrix}, & \begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}0\\1\end{bmatrix} \end{matrix}$ $\begin{matrix} T(v)=Av=\lambda v \\ v\neq 0 \end{matrix}$ $\begin{cases} \lambda: \text{eigen value} \\ v: \ \text{〃} \ \ \text{vector} \end{cases}$ of $\begin{matrix}A\\T\end{matrix}$

(2) $A=\begin{bmatrix}3&1\\2&2\end{bmatrix}$, 求 $\begin{cases} \lambda: (A-\lambda I)v=0 \Rightarrow (A-\lambda I) \text{ singular} \Rightarrow \det(A-\lambda I)=0 \quad \begin{cases}\lambda_1=4\\ \lambda_2=1\end{cases} \\ v: \begin{cases} \lambda_1=4 \quad \begin{bmatrix}-1&1\\2&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow v_1=\begin{bmatrix}1\\1\end{bmatrix} \quad \left(\det\left(\begin{bmatrix}3-\lambda & 1\\ 2 & 2-\lambda\end{bmatrix}\right)=\lambda^2-5\lambda+4\right) \\ \lambda_2=1 \quad \begin{bmatrix}2&1\\2&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow v_2=\begin{bmatrix}-1\\2\end{bmatrix} \quad P=\begin{bmatrix}1&-1\\1&2\end{bmatrix}, P^{-1}AP=\begin{bmatrix}4&0\\0&1\end{bmatrix} \end{cases} \end{cases}$

(3) $A=P\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}P^{-1} \Rightarrow A^k=P\begin{bmatrix}\lambda_1&\\&\lambda_2\end{bmatrix}^kP^{-1}=P\begin{bmatrix}\lambda_1^k&\\&\lambda_2^k\end{bmatrix}P^{-1}$

(4) $x_k=Ax_{k-1} \Rightarrow x_k=A^2x_{k-2}=\cdots=A^kx_0=P\begin{bmatrix}\lambda_1^k&\\&\lambda_2^k\end{bmatrix}\underline{P^{-1}x_0}=[v_1\ v_2]\begin{bmatrix}\lambda_1^k&\\&\lambda_2^k\end{bmatrix}\underline{\begin{bmatrix}c_1\\c_2\end{bmatrix}}$

$$=\underline{c_1\lambda_1^kv_1+c_2\lambda_2^kv_2}$$

$\begin{cases} a_k=3a_{k-1}+b_{k-1}, \\ b_k=2a_{k-1}+2b_{k-1}, \end{cases}$ $k\geq 1$, $\begin{cases} a_0=3 \\ b_0=9 \end{cases}$

$\underset{x_k}{\begin{bmatrix}a_k\\b_k\end{bmatrix}}=\underset{A}{\begin{bmatrix}3&1\\2&2\end{bmatrix}}\underset{x_{k-1}}{\begin{bmatrix}a_{k-1}\\b_{k-1}\end{bmatrix}}$, $\underset{x_0}{\begin{bmatrix}a_0\\b_0\end{bmatrix}}=\begin{bmatrix}3\\9\end{bmatrix}$ $\Rightarrow x_k=\begin{bmatrix}a_k\\b_k\end{bmatrix}=\begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}4^k&0\\0&1^k\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}=\underline{\underline{5\cdot4^k\begin{bmatrix}1\\1\end{bmatrix}+2\cdot1^k\begin{bmatrix}-1\\2\end{bmatrix}}}$

$x_k = A\ x_{k-1}$, $x_0=\begin{bmatrix}3\\9\end{bmatrix}$

應用

(1) 到 $V = \text{span}(v_1, v_2)$ 的投影矩陣 (Example 1)

$\mathcal{B} = \{v_1, v_2, a\}$, $a \in V^{\perp}$

$$P_V = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}P^{-1}$$

$$\begin{tikzcd} \mathcal{E}: \mathbb{R}^n \arrow[r, "A"] \arrow[d, "P^{-1}"'] & \mathbb{R}^n \\ \mathcal{B}: \mathbb{R}^n \arrow[r, "A'"] & \mathbb{R}^n \arrow[u, "P"'] \end{tikzcd}$$

$$A = P A' P^{-1}$$

(2) 繞 a 旋轉 θ 角 (Example 2)

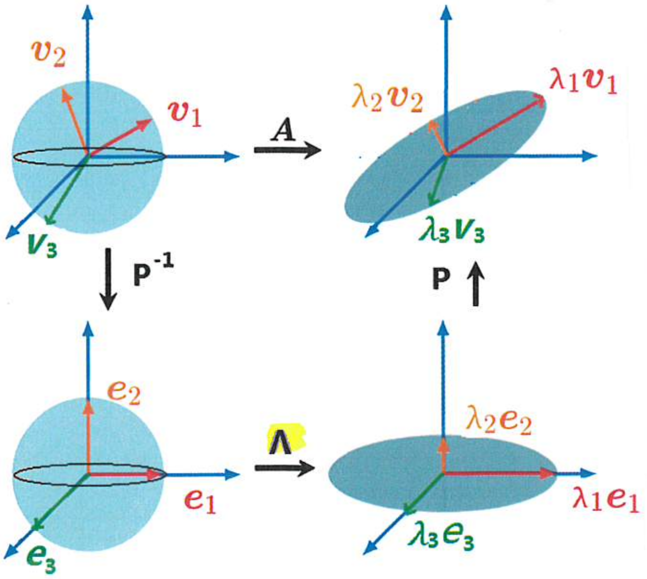
$\mathcal{B} = \{v_1, v_2, a\}$, $v_1, v_2 \in V = a^{\perp}$

$$R_\theta = P\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1}$$

(3) Eigenvalues decomposition

$\mathcal{B} = \{v_1, \cdots, v_n\}$ v_i : eigen vector

$$A = P\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}P^{-1} \quad \begin{cases} Av_i = \lambda_i v_i \\ \text{(作用方向不變)} \end{cases}$$



(4) Orthogonal decomposition

$\mathcal{B} = \{q_1, \cdots, q_n\}$ orthonormal eigen basis

$$A = Q\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}Q^T \quad \begin{cases} A: \text{symmetric} \\ \Rightarrow \text{Quadratic form, Constraint optimization, PCA} \end{cases}$$

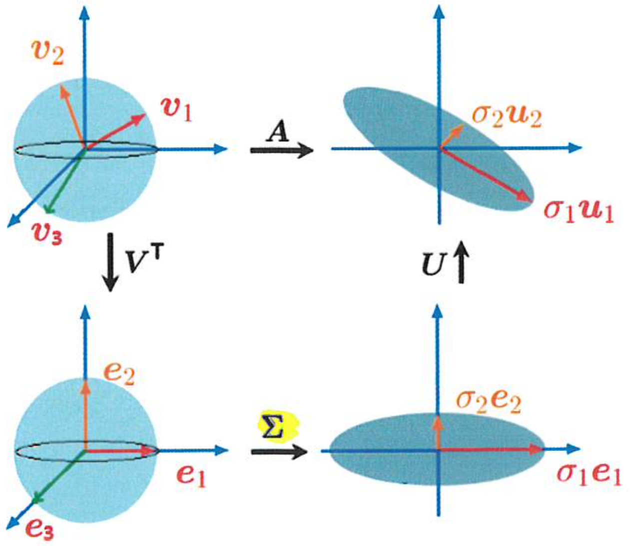
(5) Singular-values decomposition (SVD)

$\mathcal{U}$: orthonormal basis for $\mathbb{R}^m$

$\mathcal{V}$ 〃 〃 $\mathbb{R}^n$

$$A_{m\times n} = U \Sigma V^T$$

$$\begin{cases} v_i: \text{最大作用方向} \\ \Rightarrow \text{影像壓縮} \end{cases}$$



(6) Jordan forms

$$A = P\left[\begin{array}{ccc|cc|cc} \lambda_1 & 1 & & & & & \\ & \lambda_1 & 1 & & & & \\ & & \lambda_1 & & & & \\ \hline & & & \lambda_1 & 1 & & \\ & & & & \lambda_1 & & \\ \hline & & & & & \lambda_2 & 1 \\ & & & & & & \lambda_2 \end{array}\right]P^{-1}$$

- **Example 1** $V=\{x\in\mathbb{R}^3 \mid x-2y-z=0\}$, $P_V, P_a, H_V=?$ H_V: 对V作鏡射.

$$a=\begin{bmatrix}1\\-2\\-1\end{bmatrix},\quad V=\text{span}(\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{v_1},\underbrace{\begin{bmatrix}1\\1\\-1\end{bmatrix}}_{v_2}),$$

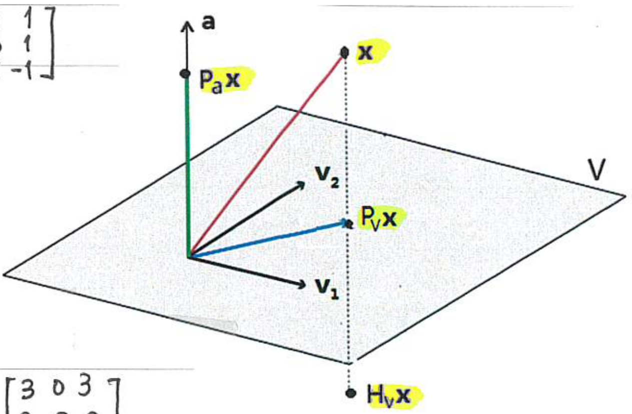
$$\begin{cases}P_V+P_a=I_3\\ P_V-P_a=H_V\end{cases}$$

(A) 基底 $\mathcal{E}=\{e_1,e_2,e_3\}$

(1) $P_V=A(A^TA)^{-1}A^T=\frac{1}{6}\begin{bmatrix}5&2&1\\2&2&-2\\1&-2&5\end{bmatrix}$, $A=\begin{bmatrix}1&1\\0&1\\1&-1\end{bmatrix}$

(2) $P_a=\frac{1}{\|a\|^2}aa^T=\frac{1}{6}\begin{bmatrix}1&-2&-1\\-2&4&2\\-1&2&1\end{bmatrix}$

(3) $H_V=P_V-P_a=\frac{1}{6}\begin{bmatrix}4&4&2\\4&-2&-4\\2&-4&4\end{bmatrix}$



(B) 基底 $\mathcal{B}=\{v_1,v_2,a\}$, $P=\begin{bmatrix}1&1&1\\0&1&-2\\1&-1&-1\end{bmatrix}$, $P^{-1}=\frac{1}{6}\begin{bmatrix}3&0&3\\2&2&-2\\1&-2&-1\end{bmatrix}$

$$\begin{cases}P_V'(v_1)=v_1\\ P_V'(v_2)=v_2\\ P_V'(a)=0\end{cases}$$

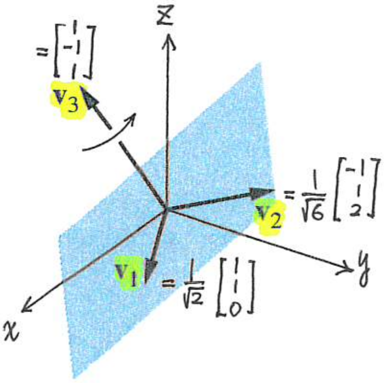
(1) $P_V'=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\Rightarrow P_V=PP_V'P^{-1}=\frac{1}{6}\begin{bmatrix}5&2&1\\2&2&-2\\1&-2&5\end{bmatrix}$

(2) $P_a'=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\Rightarrow P_a=PP_a'P^{-1}=\frac{1}{6}\begin{bmatrix}1&-2&-1\\-2&4&2\\-1&2&1\end{bmatrix}$

(3) $H_V'=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\Rightarrow H_V=PH_V'P^{-1}=\frac{1}{6}\begin{bmatrix}4&4&2\\4&-2&4\\2&-4&4\end{bmatrix}$

- **Example 2** $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ rotate around $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ by $\theta=\frac{2}{3}\pi$

$\mathcal{B}=\{v_1,v_2,v_3\}$, rotate around v_3 by $\theta=\frac{2}{3}\pi$



$$A'=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}&0\\\frac{\sqrt3}{2}&-\frac{1}{2}&0\\0&0&1\end{bmatrix}$$

$$P=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt6}&1\\\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&-1\\0&\frac{2}{\sqrt6}&1\end{bmatrix}$$

$$R_\theta=PA'P^{-1}=\begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix}=\overset{\text{繞}x\text{軸}\frac{\pi}{2}}{\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}}\overset{\text{繞}z\text{軸}\frac{\pi}{2}}{\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}}$$

$$\begin{bmatrix}-y\\-z\\x\end{bmatrix}\longleftarrow\begin{bmatrix}-y\\x\\z\end{bmatrix}\longleftarrow\begin{bmatrix}x\\y\\z\end{bmatrix}$$

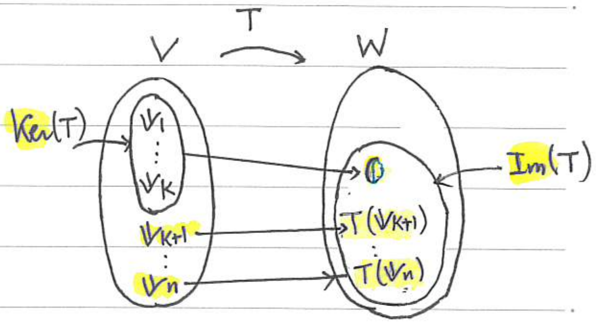
Linear Transformations on Abstract Vector Spaces

$\left\{\begin{matrix} V \\ W \end{matrix}\right.$ (abstract) Vector spaces, $T: V \longrightarrow W$ 線性變換 : $\left\{\begin{matrix} T(u+v) = T(u)+T(v) \\ T(cu) = c\,T(u) \end{matrix}\right. \quad \forall \begin{matrix} u,v \in V \\ c \in \mathbb{R} \end{matrix}$

定理 $T \in L(V,W) = \{\mathcal{T}: V \to W \mid \mathcal{T}: \text{linear}\}$ $\quad \left\{\begin{matrix} \dim V = n \\ \dim W = m \end{matrix}\right.$ 基底 $\begin{matrix} \mathcal{V} = \{v_1, \dots, v_n\} \\ \mathcal{W} = \{w_1, \dots, w_m\} \end{matrix}$

(1) $\left\{\begin{matrix} \text{Ker}(T) = \{x \in V \mid T(x) = 0\} \lhd V \\ \text{Im}(T) = \{T(x) \mid x \in V\} \lhd W \end{matrix}\right.$

(2) $\dim \text{Ker}(T) + \dim \text{Im}(T) = n$



(3) $\left\{\begin{matrix} 0 \in \text{Ker}(T) \\ \text{Ker}(T) = \{0\} \iff T: 1\text{-}1 \end{matrix}\right.$

(4) $\text{rank}(A) = m \Rightarrow T: \text{onto}$

證 (2) 設 $\dim \text{Ker}(T) = k$, 基底為 $\{v_1, \dots, v_K\}$ $\overset{?}{\Rightarrow} \{T(v_{K+1}), \dots, T(v_n)\}$ Im(T) 基底
擴充 $\{v_1, \dots, v_K\} \cup \{v_{K+1}, \dots, v_n\}$ 為 V 之基底

(甲)(生成) $\forall w \in \text{Im}(T), \exists v = \sum_{i=1}^{n} c_i v_i$, $w = T(v) = \sum_{i=k+1}^{n} c_i T(v_i)$

(乙)(獨立) $\sum_{i=K+1}^{n} c_i T(v_i) = 0 \Rightarrow T\left(\sum_{i=K+1}^{n} c_i v_i\right) = 0 \Rightarrow \sum_{i=K+1}^{n} c_i v_i \in \text{Ker}(T)$

$\Rightarrow \sum_{i=K+1}^{n} c_i v_i = \sum_{i=1}^{K} c_i v_i \Rightarrow c_i = 0, \ 1 \le i \le n$

$\mathcal{P}_k = \{a_0 + a_1 t + \dots + a_k t^k \mid a_i \in \mathbb{R}\}$

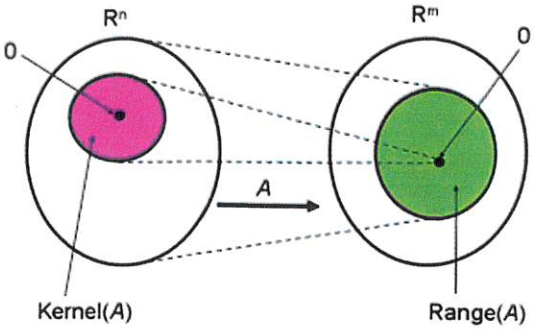
Example

	Ker(T)	Im(T)
(a) $D: \mathcal{P}_3 \to \mathcal{P}_2$ $f \mapsto f'$	$\mathcal{P}_0$	$\mathcal{P}_2$
(b) $M: \mathcal{C}^0([1,2]) \to \mathcal{C}^0([1,2])$ $f \longmapsto tf$	$\{0\}$ $tf(t) \equiv 0 \Rightarrow f(t) \equiv 0,$	$\mathcal{C}^0([1,2])$ $\frac{f(t)}{t} \longmapsto f(t)$
(c) $\mathcal{T}: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ $f \mapsto [f(0)\ f(1)]$	$\{at(t-1) \mid a \in \mathbb{R}\}$	$\mathbb{R}^2$ $a + (b-a)t \mapsto [a\ b]$
(d) $\mathcal{S}: \mathcal{P}_2 \longrightarrow \mathbb{R}^3$ $f \longmapsto [f(0)\ f(1)\ f(2)]$	$\{0\}$	$\mathbb{R}^3$ (Lagrange Int.) $\frac{a}{2}(t-1)(t-2) - b\,t(t-2) + \frac{c}{2}\,t(t-1)$ $\mapsto [a\ b\ c]$
(e) $\mathcal{S}': \mathcal{P}_1 \longrightarrow \mathbb{R}^3$ $f \longmapsto [f(0)\ f(1)\ f(2)]$	$\{0\}$ $\left\{\begin{matrix} p = a \\ p+q = b \\ p+2q = c \end{matrix}\right. \Leftarrow$	$\{[a\ b\ c] \mid a-2b+c=0\}$ $p+qt \mapsto [a\ b\ c]$

$A_{m\times n}$, $T = \mu_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\begin{cases} Ker(T) = N(A) \\ Im(T) = C(A) \end{cases}$

- 定理 4.13 (1) $\begin{cases} \dim N(A) + \dim C(A) = n \\ \dim Ker(T) + \dim Im(T) = n \end{cases}$

  (2) $rank(A) = \dim C(A) = \dim Im(T) \leq n$



- 定理4.14 $A_{m\times n}$, $B_{n\times p}$

  (1) $rank(AB) \leq rank(A)$

  (2) $n=p$, B nonsingular $\Rightarrow rank(AB) = rank(A)$ $rank(B) = n$, B: 滿秩

  (3) $rank(AB) \leq rank(B)$

  (4) $m=n$, A nonsingular $\Rightarrow rank(AB) = rank(B)$

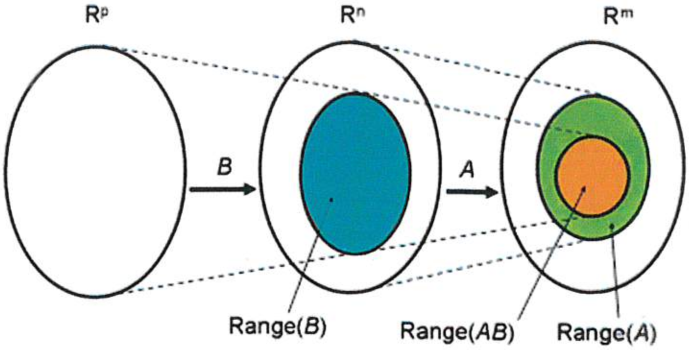
證

(1) $rank(A) \geqq rank(AB)$

(2) $rank(AB) \geqq rank(ABB^{-1}) = rank(A)$

(3) $A : Im(B) \longrightarrow \mathbb{R}^m$

定理4.13 (2) $\Rightarrow$



Change of Basis formula

- vector space V, 基底 $\mathcal{B}=\{v_1,\dots,v_n\}$, $x = x_1v_1+\cdots+x_nv_n \in V$, $\overset{座標}{C_{\mathcal{B}}(x)} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$
- $T: V \longrightarrow W$ 線性變換, $\begin{cases} V \\ W \end{cases}$ 基底 $\begin{matrix} \mathcal{V}=\{v_1,\dots,v_n\}, & \mathcal{V}'=\{v_1',\dots,v_n'\} \\ \mathcal{W}=\{w_1,\dots,w_m\}, & \mathcal{W}'=\{w_1',\dots,w_m'\} \end{matrix}$

$$A = [T]_{\mathcal{V},\mathcal{W}} = \begin{bmatrix} | & & | & & | \\ C_{\mathcal{W}}(T(v_1)) & \cdots & C_{\mathcal{W}}(T(v_j)) & \cdots & C_{\mathcal{W}}(T(v_n)) \\ | & & | & & | \end{bmatrix}, \begin{cases} T(v_j) = a_{1j}w_1+\cdots+a_{mj}w_m \\ a_j = C_{\mathcal{W}}(T(v_j)) = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \end{cases}$$

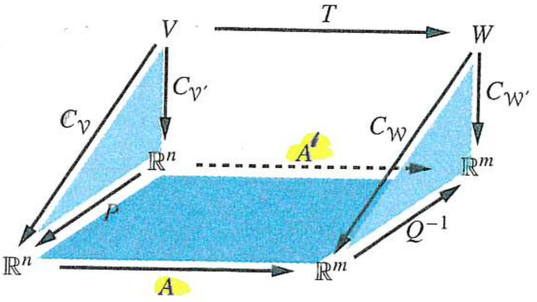
Example $D: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$, $\overset{\text{basis}}{\begin{cases} \mathcal{V}=\{1,t,t^2,t^3\}, & \mathcal{V}'=\{1,t-1,(t-1)^2,(t-1)^3\} \\ \mathcal{W}=\{1,t,t^2\} & \mathcal{W}'=\{1,t-1,(t-1)^2\} \end{cases}}$
$\quad\quad f \longmapsto f'$

$$\begin{tikzcd}
& f(t) = 2-t+5t^2+4t^3 = 10+21(t-1)+17(t-1)^2+4(t-1)^3 \arrow[r, "D"] \arrow[d, "C_{\mathcal{V}'}"'] & f'(t) = -1+10t+12t^2 = 21+34(t-1)+12(t-1)^2 \arrow[d, "C_{\mathcal{W}'}"] & \\
\mathcal{V}'=\{1,t-1,(t-1)^2,(t-1)^3\}, & C_{\mathcal{V}'}(f) = \begin{bmatrix} 10 \\ 21 \\ 17 \\ 4 \end{bmatrix} \arrow[r, "A'=\begin{bmatrix} 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{bmatrix}"] \arrow[d, "P=\begin{bmatrix} 1&-1&1&-1 \\ 0&1&-2&3 \\ 0&0&1&-3 \\ 0&0&0&1 \end{bmatrix}"'] & \begin{bmatrix} 21 \\ 34 \\ 12 \end{bmatrix} = C_{\mathcal{W}'}(f') \arrow[d, "Q=\begin{bmatrix} 1&-1&1 \\ 0&1&-2 \\ 0&0&1 \end{bmatrix}"] & \mathcal{W}'=\{1,t-1,(t-1)^2\} \\
\mathcal{V}=\{1,t,t^2,t^3\} \quad C_{\mathcal{V}} \arrow[r] & C_{\mathcal{V}}(f) = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 4 \end{bmatrix} \arrow[r, "A=\begin{bmatrix} 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{bmatrix}"] & \begin{bmatrix} -1 \\ 10 \\ 12 \end{bmatrix} = C_{\mathcal{W}}(f') & C_{\mathcal{W}} \arrow[l] \quad \mathcal{W}=\{1,t,t^2\}
\end{tikzcd}$$

定理

$$\begin{cases} P = \begin{bmatrix} | & & | & & | \\ C_{\mathcal{V}}(v_1'), & \dots, & C_{\mathcal{V}}(v_j'), & \dots, & C_{\mathcal{V}}(v_n') \\ | & & | & & | \end{bmatrix} & (\mathcal{V}' \to \mathcal{V}) \\ Q = \begin{bmatrix} | & & | & & | \\ C_{\mathcal{W}}(w_1'), & \dots, & C_{\mathcal{W}}(w_j'), & \dots, & C_{\mathcal{W}}(w_m') \\ | & & | & & | \end{bmatrix} & (\mathcal{W}' \to \mathcal{W}) \end{cases}$$

$$\begin{cases} A = [T]_{\mathcal{V},\mathcal{W}} \\ A' = [T]_{\mathcal{V}',\mathcal{W}'} \end{cases} \Longrightarrow \begin{cases} A' = Q^{-1}AP \\ A = QA'P^{-1} \end{cases}$$



Example 2 $T : \mathcal{M}_{2\times2} \longrightarrow \mathcal{M}_{2\times2}$

$$X \longmapsto MX, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix} \qquad (R_{12})$$

$\mathcal{M}_{2\times2}$: 基底

$\cdot\ \mathcal{V}' = \left\{ \underset{v_1'}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_2'}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{v_3'}{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}}, \underset{v_4'}{\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}} \right\}$

$\cdot\ \mathcal{V} = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, "A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}"] \arrow[d, swap, "P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}"] & \mathbb{R}^4 \arrow[d, "Q = P"] \\ \mathbb{R}^4 \arrow[r, "A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}"] & \mathbb{R}^4 \end{tikzcd}$$

$$A' = Q^{-1}AP$$