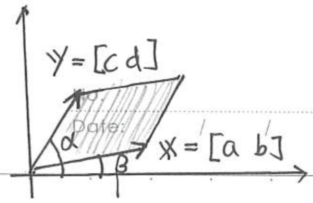
Chapter 5 Determinants

Determinants (行列式)



$$\begin{cases} \det : & M_{n\times n} \longrightarrow \mathbb{R} \\ D : & \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R} \end{cases}$$

$$\text{▨} = \|x\|\|y\| \sin(\alpha-\beta) = \|x\|\|y\|(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = ad-bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

(signed volume)

(A) Recurrence

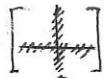
(1) $\det([a]) = D([a]) = a$ =

(2) $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = D([a\ b], [c\ d]) = ad-bc$ = 

(3) $\det \begin{bmatrix} a & b & c \\ u & v & w \\ m & n & p \end{bmatrix} = D([a\ b\ c], [u\ v\ w], [m\ n\ p]) = a\begin{vmatrix} v & w \\ n & p \end{vmatrix} - b\begin{vmatrix} u & w \\ m & p \end{vmatrix} + c\begin{vmatrix} u & v \\ m & n \end{vmatrix}$ = 

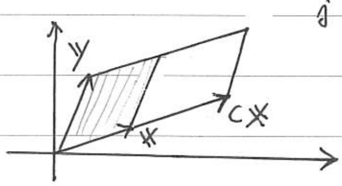
$\vdots$

(n) $\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = D(A_1, A_2, \ldots, A_n) = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots \pm a_{1n}\det A_{1n}$

$A_{ij} = i \to$  $\uparrow j$

(或 $\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$)

(B) Axioms: $(n=2,\ \forall x, y, z \in \mathbb{R}^2)$

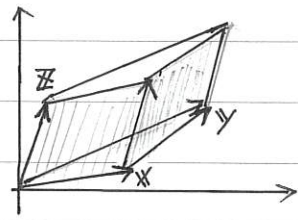


(D1) $D([1\ 0], [0\ 1]) = D(e_1, e_2) = 1$

(D2) $D(cx, y) = c\,D(x, y)$

(D3) $D(x+y, z) = D(x, z) + D(y, z)$

(D4) $D(x, y) = -D(y, x)$



Property

(1) $D(z, x+y) = D(z, x) + D(z, y)$ 証 (D3)+(D4)

(2) $D(x, cy) = c\,D(x, y)$ (D2)+(D4)

(3) $D(0, y) = D(x, 0) = 0$ (D2) $c=0$

(4) $\forall x, y\ \ D(x, y) = -D(y, x) \iff \forall x,\ D(x, x) = 0$

"$\Rightarrow$" $D(x,x) = -D(x,x)$

"$\Leftarrow$" $0 = D(x+y, x+y) = D(x,y) + D(y,x)$

(5) $D(x, cx+y) = D(x, y)$

定理1 D: 滿足 (D1) – (D4) $\iff D([a\ b], [c\ d]) = ad-bc$

証 "$\Leftarrow$"

"$\Rightarrow$" (甲) $L = D(ae_1 + be_2,\ ce_1 + de_2) = ad\,D(e_1, e_2) + bc\,D(e_2, e_1) = ad - bc$

(乙) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & -\frac{c}{a}b + d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & -\frac{c}{a}b + d \end{vmatrix} = ad - bc$

$(a \neq 0)$

定理2 存在唯一 $D: \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ 滿足 $(\forall\ 1 \le i \le n)$

(D1) $D(e_1, e_2, \dots, e_n) = 1$

(D2) $D(A_1, \dots, cA_i, \dots, A_n) = c\,D(A_1, \dots, A_i, \dots, A_n)$

(D3) $D(A_1, \dots, A_i + B_i, \dots, A_n) = D(A_1, \dots, A_i, \dots, A_n) + D(A_1, \dots, B_i, \dots, A_n)$

(D4) $D(A_1, \dots, A, A, \dots, A_n) = 0 \quad (A_i = A_{i+1})$

定義 $\det: \mathcal{M}_{n\times n} \to \mathbb{R}$, $\det(A) := D(A_1, \dots, A_n) = |a_{ij}|$

Proposition

(D4) $D(\dots, A, A, \dots) = 0$ (相鄰兩列相等)

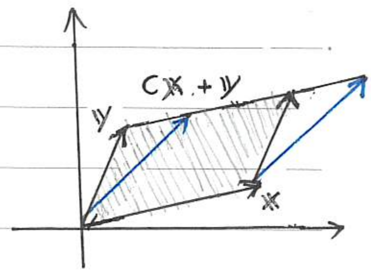
$\Leftrightarrow$ (D5) $D(\dots, A, B, \dots) = -D(\dots, B, A, \dots)$ (〃 〃 互換)

$\Leftrightarrow$ (D6) $D(\dots, A, \dots, A, \dots) = 0$ (任意 〃 相等)

$\Leftrightarrow$ (D7) $D(\dots, A, \dots, B, \dots) = -D(\dots, B, \dots, A, \dots)$ (〃 〃 互換)

證
$$\begin{tikzcd}
(D4) \arrow[r, leftrightarrow] \arrow[d, shift left, leftarrow] \arrow[d, shift right] & (D5) \arrow[d, shift left, leftarrow] \arrow[d, shift right, "* \text{ Ex.}"'] \\
(D6) \arrow[r, leftrightarrow] & (D7)
\end{tikzcd}$$

$(D4) + (D5) \Rightarrow (D6)$



Proposition (D8) $D(\dots, 0, \dots) = 0$

(D9) $D(\dots, A_i, \dots, A_j, \dots) = D(\dots, A_i, \dots, cA_i + A_j, \dots)$

Example

$$\det\begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 4 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 4 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 12 \end{vmatrix} = -12\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = -12\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= -12$$

Proposition 基本列運算*: $A \rightsquigarrow A' = EA$, 則 $\det(A') = \det(EA) = \det(E)\det(A)$

證

R_{24}: $\det(A') = -\det(A)$, $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $\det(E) = -1$,

cR_2: $\det(A') = c\det(A)$, $E = \begin{bmatrix} 1 & & & \\ & c & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$, $\det(E) = c$

$cR_2 + R_4$: $\det(A') = \det(A)$, $E = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & c & & 1 \end{bmatrix}$, $\det(E) = 1$

註: $\det(E^T) = \det(E)$

定理3 $A_{n\times n}$ nonsingular $\Longleftrightarrow$ $\det(A)\neq 0$

証 $\begin{cases} A \rightsquigarrow I_n, & \det(A)\neq 0 \\ A \rightsquigarrow \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & 0 \end{bmatrix}, & \det(A)=0 \end{cases}$

Proposition $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & & \vdots \\ & & & a_{nn} \end{bmatrix}$, $\det(A)=a_{11}a_{22}\cdots a_{nn}$,

証: $\begin{cases} \exists\, a_{ii}=0, & A\text{: singular} \\ \text{否則} & \det(A)=a_{11}\cdots a_{nn}\begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} \end{cases}$

Proposition $\det(AB)=\det(A)\det(B)$ $\begin{bmatrix} -A_1- \\ \vdots \\ -A_n- \end{bmatrix}[B]=\begin{bmatrix} -A_1B- \\ -A_nB- \end{bmatrix}$

証 (甲) A: singular $\Rightarrow c_1A_1+\cdots+c_nA_n=0 \Rightarrow c_1A_1B+\cdots+c_nA_nB=0 \Rightarrow AB$ singular

(乙) A: nonsingular $\Rightarrow \begin{cases} A=E_1\cdots E_k, & \det(A)=\det(E_1)\cdots\det(E_k) \\ AB=E_1\cdots E_kB, & \det(AB)=\det(E_1)\cdots\det(E_k)\det(B) \end{cases}$

Proposition (1) A: nonsingular $\Rightarrow \det(A^{-1})=\dfrac{1}{\det(A)}$ 証 $A^{-1}A=I_n$

(2) $B=P^{-1}AP \Rightarrow \det(B)=\det(A)$

Proposition $\det(A^T)=\det(A)$ (∵行適用 D1−D9)

証 $\begin{cases} A \text{ singular} \Rightarrow A^T \text{ singular} \\ A=E_1\cdots E_k \Rightarrow A^T=E_k^T\cdots E_1^T \end{cases}$

Proposition $T:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ (linear), $T=\mu_A$

$$\Rightarrow D(T(v_1),\cdots,T(v_n))=\det\begin{bmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{bmatrix}=\det\left(A\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}\right)=\det(A)\det\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

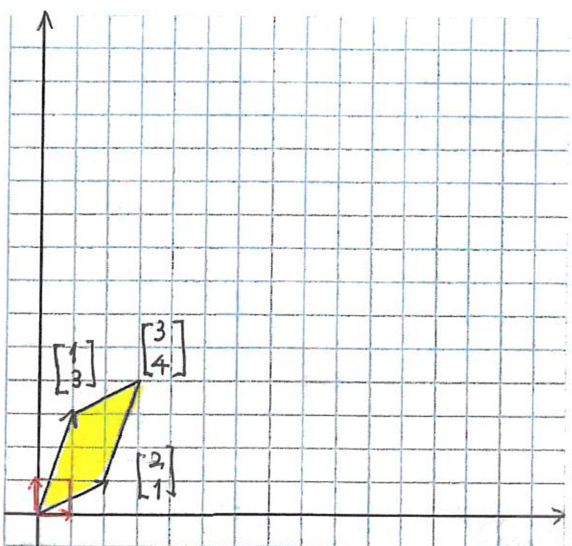
(面積放大率)

Example

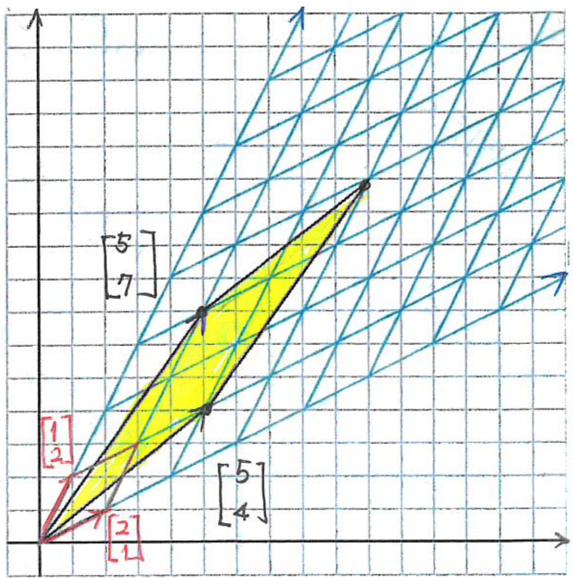
$$A\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}=\begin{bmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{bmatrix}$$

$$\begin{array}{cc} a_1 & a_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \end{array}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}=\begin{array}{cc} 2a_1+a_2 & a_1+3a_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 5 & 5 \\ 4 & 7 \end{bmatrix}} \end{array}$$

det: $3 \cdot 5 = 15$



$T=\mu_A$ $\longrightarrow$



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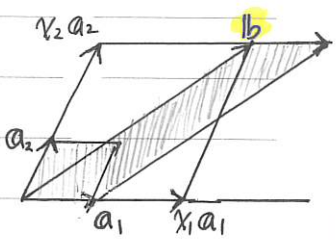


Proposition (Cramer) $\begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbb{b} \Rightarrow x_1 = \dfrac{D(\mathbb{b}, a_2, a_3)}{\underset{\det(A)}{\underset{\|}{D(a_1, a_2, a_3)}}}$

証 $D(\mathbb{b}, a_2, a_3) = D(x_1 a_1 + x_2 a_2 + x_3 a_3, a_2, a_3)$

$= x_1 D(a_1, a_2, a_3)$

$$\left\{\begin{array}{l} \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbb{b} \\ x_2 = \dfrac{D(a_1, \mathbb{b})}{D(a_1, a_2)} \end{array}\right.$$

Proposition A: nonsingular $\Rightarrow A^{-1} = [x_{ij}]$, $x_{ij} = \dfrac{C_{ji}}{\det(A)}$

証: (甲) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left\{\begin{array}{l} x_{23} = \dfrac{1}{\det(A)}\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 1 & a_{33} \end{vmatrix} = \dfrac{C_{32}}{\det(A)} \\ x_{32} = \dfrac{1}{\det(A)}\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 0 \end{vmatrix} = \dfrac{C_{23}}{\det(A)} \end{array}\right.$$

(餘因子)
Cofactor $C_{ij} = (-1)^{i+j}\det(A_{ij})$

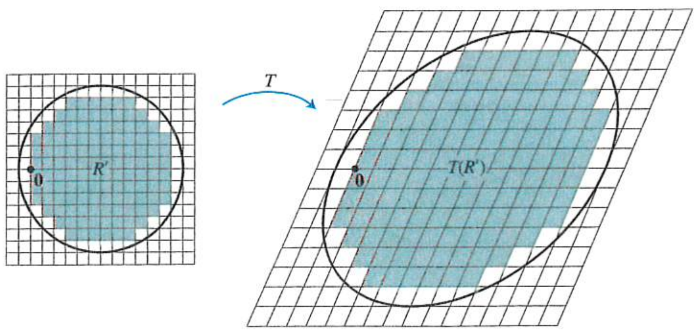
Cofactor matrix $C = [C_{ij}]$

$\Rightarrow A^{-1} = \dfrac{1}{\det(A)} C^T$

(乙) $AC^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \overset{*}{\det(A)} & 0 & 0 \\ 0^{**} & \det(A) & 0 \\ 0^{***} & 0 & \det(A) \end{bmatrix} = \det(A) I_3$

$$\left\{\begin{array}{l} * = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \det(A) \\ ** = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \\ *** = a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13} = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \end{array}\right.$$

(續前頁) 一般圖形 $\Rightarrow$ 放大 $\det(A)$ 倍



$$\begin{cases} (D1): & D(e_1, e_2, \dots, e_n) = 1 \\ (D2): & D(\dots, \alpha A, \dots) = \alpha\, D(\dots, A, \dots) \qquad \forall\ 1\le i\le n \\ (D3): & D(\dots, A+B, \dots) = D(\dots, A, \dots) + D(\dots, B, \dots) \\ (D4): & D(\dots, A, A, \dots) = 0 \end{cases}$$

Proposition $D: \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ (det) 滿足 (D1) − (D4) $\iff \begin{cases} \det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} & (1\le i\le n)\ (i\text{列展開}) \\ = \sum_{i=1}^{n} a_{ij} C_{ij} & (1\le j\le n)\ (j\text{行}\ 〃\ ) \end{cases}$

證: "$\Rightarrow$" $(i=1)$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\det\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{12}\det\begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{matrix}(D2)\\(D3)\end{matrix}$$

(相同列運算)

$$= a_{11}\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} + a_{12}\det\begin{bmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$$

$$= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33},$$

"$\Leftarrow$" (Induction)

(甲) $n=2$ 成立 (定理1)

(乙) $n-1\ (=3)$ 成立 (擬証 $n\ (=4)$ 成立: $\det(A) \overset{j=1}{=\!=} a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + a_{31}\det(A_{31}) - a_{41}\det(A_{41})$)

$$\begin{cases} (D1): & \det\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} = 1\cdot\det(I_3) + 0\cdot C_{21} + 0\cdot C_{31} + 0\cdot C_{41} = 1 \\ (D2): & \det\begin{bmatrix} a&m&n&p\\b&r&s&t\\\alpha c&\alpha u&\alpha v&\alpha w\\d&x&y&z \end{bmatrix} = a(\alpha C_{11}) + b(\alpha C_{21}) + (\alpha c)C_{31} + d(\alpha C_{41}) = \alpha\det(A) \\ (D3): & \\ (D4): & \det\begin{bmatrix} a&m&n&p\\b&u&v&w\\b&u&v&w\\d&x&y&z \end{bmatrix} = a\cdot 0 - b\cdot\det\begin{bmatrix} m&n&p\\u&v&w\\x&y&z \end{bmatrix} + b\cdot\det\begin{bmatrix} m&n&p\\u&v&w\\x&y&z \end{bmatrix} - d\cdot 0 = 0 \end{cases}$$

定理2 存在唯一 $D: \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ 滿足 (D1) − (D4)

證 (1) 存在 ↗

(2) 唯一: $A \rightsquigarrow$ 相同 reduced echelon matrix $\Rightarrow \det(A)$ 相同.