Chapter 5  Determinants
Determinants (行列式)

\[
\begin{align*}
\textbf{det} : & \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \\
\mathbf{D} : & \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \rightarrow \mathbb{R}
\end{align*}
\]

(A) 随回数
(1) \( \text{det}([a]) = \mathbf{D}( [a] ) = a \)
(2) \( \text{det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{D}( [a \ b] , [c \ d] ) = ad - bc \)
(3) \( \text{det} \begin{bmatrix} a & b & c \\ u & v & w \\ m & n & p \end{bmatrix} = \mathbf{D}( [a \ b \ c] , [u \ v \ w] , [m \ n \ p] ) = a \\begin{vmatrix} v & w \\ m & p \end{vmatrix} - b \begin{vmatrix} u & w \\ m & n \end{vmatrix} + c \begin{vmatrix} u & v \\ n & p \end{vmatrix} \)

\( \vdots \)

(6) \( \text{det} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = \mathbf{D}(A_1, A_2, \ldots, A_n) = a_{11} \cdot \text{det} A_{11} - a_{12} \cdot \text{det} A_{12} + \ldots \pm a_{1n} \cdot \text{det} A_{1n} \)

\( A_{ij} = i \rightarrow \left[ \begin{array}{c}
 1 \\
 c \end{array} \right] \)

(B) Axioms: \( (n=2, \forall x, y, z \in \mathbb{R}^2) \)

(D1) \( \mathbf{D}( [0 \ 0] , [1 \ 1] ) = \mathbf{D}(\emptyset, \emptyset) = 1 \)
(D2) \( \mathbf{D}(c, y) = c \mathbf{D}(x, y) \)
(D3) \( \mathbf{D}(x+y, z) = \mathbf{D}(x, z) + \mathbf{D}(y, z) \)
(D4) \( \mathbf{D}(x, y) = -\mathbf{D}(y, x) \)

Property
(1) \( \mathbf{D}(z, x+y) = \mathbf{D}(z, x) + \mathbf{D}(z, y) \) \( \stackrel{\text{证}}{=} \text{(D3)+(D4)} \)
(2) \( \mathbf{D}(x, cy) = c \mathbf{D}(x, y) \) \( \stackrel{\text{证}}{=} \text{(D2)+(D4)} \)
(3) \( \mathbf{D}(0, y) = \mathbf{D}(x, 0) = 0 \) \( \stackrel{\text{证}}{=} \text{(D2)} \)
(4) \( \forall x, y \mathbf{D}(x, y) = \mathbf{D}(y, x) \)
(5) \( \mathbf{D}(x, c \times y) = \mathbf{D}(x, y) \)

定理 1: \( \exists \text{D满足(D1)-(D4)} \iff \mathbf{D}( [a \ b] , [c \ d] ) = ad - bc \)

证 "\( \Rightarrow \)"
1. \( L = \mathbf{D}(a \epsilon_1 + b \epsilon_2, c \epsilon_1 + d \epsilon_2) = ad \mathbf{D}(\emptyset, \emptyset) + bc \mathbf{D}(\emptyset, \emptyset) = ad - bc \)
2. \( \left| \begin{array}{cc}
 a & b \\
 c & d \\
 \end{array} \right| = ad - bc \)

(\( a \neq 0 \))
定理 2. 存在唯 D: \( \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} \) 满足

\[
(D1) \quad D(e_1, e_2, \ldots, e_n) = 1
\]
\[
(D2) \quad D(A_1, \ldots, cA_i, \ldots, A_n) = c \cdot D(A_1, \ldots, A_i, \ldots, A_n)
\]
\[
(D3) \quad D(A_i, A_i + B, \ldots, A_n) = D(A_1, \ldots, A_i, \ldots, A_n) + D(A_1, \ldots, B, \ldots, A_n)
\]
\[
(D4) \quad D(A_1, \ldots, A_i, A_i, \ldots, A_n) = 0 \quad (A_i = A_i +)
\]

定義. \( \det: \mathbb{R}^{n \times n} \to \mathbb{R} \), \( \det(A) := D(A_1, \ldots, A_n) = \left| a_{ij} \right| \)

**Proposition**

\[
(D4) \quad D(\ldots, A, A, \ldots) = 0
\]

\(\Leftrightarrow (D5) \quad D(\ldots, A, B, \ldots) = -D(\ldots, B, A, \ldots) \quad (\text{相鄰兩行相等})
\]

\(\Leftrightarrow (D6) \quad D(\ldots, A, \ldots, A, \ldots) = 0 \quad (\text{任意任兩行}})

\(\Leftrightarrow (D7) \quad D(\ldots, A, \ldots, B, \ldots) = -D(\ldots, B, \ldots, A, \ldots) \quad (\text{相等且任兩行}})

\(\begin{align*}
(D4) & \quad \leftrightarrow (D5) \\
(D4) + (D5) & \Rightarrow (D6)
\end{align*}\)

**Proposition**

\[
D(\ldots, 0, \ldots) = 0
\]

\[
(D8) \quad D(\ldots, A_i, \ldots, A_j, \ldots) = D(\ldots, A_i, \ldots, cA_i + A_j, \ldots)
\]

**Example**

\[
\begin{vmatrix} 2 & 4 & 6 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 4 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 4 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -12
\]

\[
\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -12
\]

**Proposition**

基本列運算: \( A \rightarrow A' = EA \)，則 \( \det(A') = \det(EA) = \det(E) \det(A) \)

**証**

\[
\begin{align*}
R_2 + R_4: \quad & \det(A') = -\det(A), \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(E) = -1, \\
C_R: \quad & \det(A') = c \cdot \det(A), \quad E = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \quad \det(E) = c \\
C_R + R_4: \quad & \det(A') = \det(A), \quad E = \begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}, \quad \det(E) = 1
\end{align*}
\]

\(\therefore \quad \det(E)^n = \det(E)\)
**Proposition**

\[ A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \det(A) = a_{11}a_{22}\ldots a_{nn}. \]

- \( \exists a_{ii} = 0, \ A : \text{singular} \)
- \( \text{否则} \quad \det(A) = a_{11}a_{22}\ldots a_{nn} \)

\[ A: \text{singular} \Rightarrow C_1A_1 + \ldots + C_nA_n = 0 \Rightarrow C_1A_1B + \ldots + C_nA_nB = 0 \Rightarrow AB \text{ singular} \]

(2) \( A: \text{non-singular} \Rightarrow \begin{cases} A = E_1\cdots E_k, & \det(A) = \det(E_1)\cdots\det(E_k) \\ AB = E_1\cdots E_kB, & \det(AB) = \det(E_1)\cdots\det(E_k)\det(B) \end{cases} \)

**Proposition**

\[ \det(A^{-1}) = \frac{1}{\det(A)} \quad \text{设} \quad A^{-1}A = I_n \]

(2) \( B = P^TAP \Rightarrow \det(B) = \det(A) \)

**Proposition**

\[ \det(A^T) = \det(A) \quad \text{（行矩阵D1-D9）} \]

\[ A: \text{singular} \Rightarrow A^T: \text{singular} \]

**Proposition**

\[ T: \mathbb{R}^n \rightarrow \mathbb{R}^m \ (\text{linear}), \quad T = MA \]

\[ D(T(v_1), \ldots, T(v_n)) = \det(A) \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = \det(A) \begin{bmatrix} v_1^T & \ldots & v_n^T \end{bmatrix} \]

**Example**

\[ A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 + 3a_2 \\ 2a_1 + a_2 \\ a_1 + 3a_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & 7 \end{bmatrix} \]

\[ \det: \quad 3 \cdot 5 = 15 \]
**Proposition (Cramer)**
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & a_2 & a_3 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \mathbf{b} \Rightarrow x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}
\]

\[
D = \begin{vmatrix}
a_1 & a_2 & a_3 \\
a_2 & a_2 & a_3 \\
a_3 & a_2 & a_3
\end{vmatrix} = x_1 D(a_1, a_2, a_3)
\]

**Proposition**
\[A: \text{ nonsingular} \Rightarrow A^{-1} = [\chi_{ij}], \quad \chi_{ij} = \frac{C_{ij}}{\det(A)}\]

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
\begin{bmatrix}
\chi_{11} & \chi_{12} & \chi_{13} \\
\chi_{21} & \chi_{22} & \chi_{23} \\
\chi_{31} & \chi_{32} & \chi_{33}
\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1\end{bmatrix}
\]

\[
\chi_{13} = \frac{1}{\det(A)} \begin{vmatrix}a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = \frac{C_{13}}{\det(A)}
\]

\[
\chi_{23} = \frac{1}{\det(A)} \begin{vmatrix}a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = \frac{C_{23}}{\det(A)}
\]

\[
\chi_{32} = \frac{1}{\det(A)} \begin{vmatrix}a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = \frac{C_{32}}{\det(A)}
\]

\[
A^T C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\begin{bmatrix}
\det(A) \\
\det(A) \\
\det(A)
\end{bmatrix} = \det(A) I_3
\]

\[
\star = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = \det(A)
\]

\[
\star \star = a_{21} (C_{11} + a_{12} C_{12} + a_{23} C_{13}) = \begin{vmatrix}
a_{21} & a_{22} & a_{23} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 0
\end{vmatrix} = 0
\]

\[
\star \star \star = a_{31} (C_{11} + a_{12} C_{12} + a_{33} C_{13}) = \begin{vmatrix}
a_{31} & a_{32} & a_{33} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 0
\end{vmatrix} = 0
\]

(続前項) - 誤字補正 ⇒ 放大 \(\det(A)\) 倍
(D1): \( D(e_1, e_2, \ldots, e_n) = 1 \)
(D2): \( D(\ldots, xA, \ldots) = xD(\ldots, A, \ldots) \quad \forall 1 \leq i \leq n \)
(D3): \( D(\ldots, A + B, \ldots) = D(\ldots, A, \ldots) + D(\ldots, B, \ldots) \)
(D4): \( D(\ldots, A, A, \ldots) = 0 \)

**Proposition**

\( D: \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} \)

\[ \det(A) = \sum_{j=1}^{n} a_{i1} C_{i1} \]  
\[ = \frac{a_{i1}}{a_{i2}} \]  
\[ \frac{a_{i1}}{a_{i3}} \]  
\[ \frac{a_{i1}}{a_{i3}} \]

证：

\[ \Rightarrow \] (i = 1)

\[ \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \det \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{12} \det \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

(相同不等式)

\[ \Leftarrow \] (Induction)

(I) \( n = 2 \) 成立 (定理 1)

(II) \( n-1 (\neq 3) \) 成立 (假设 \( n-1 = 4 \) 成立：\( \det(A) = a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + a_{31} \det(A_{31}) - a_{41} \det(A_{41}) \))

\[ \begin{align*}
(D1): \ & \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \cdot \det(I_3) + 0 \cdot C_{21} + 0 \cdot C_{31} + 0 \cdot C_{41} = 1 \\
(D2): \ & \det \begin{bmatrix} a & b & c & d \\ m & n & o & p \\ r & s & t & u \\ v & w & x & y \end{bmatrix} = a(m, o, p) + b(n, o, t) + c(n, u, x) + d(o, t, u) = \chi \det(A) \\
(D3): \ & \det \begin{bmatrix} a & m & n & p \\ b & u & v & w \\ c & x & y & z \\ d & w & y & z \end{bmatrix} = a \cdot 0 - b \cdot \det \begin{bmatrix} m & n & p \\ u & v & w \\ x & y & z \end{bmatrix} + b \cdot \det \begin{bmatrix} m & n & p \\ u & v & w \\ x & y & z \end{bmatrix} - d \cdot 0 = 0 \end{align*} \]

定理 2

存在唯一 \( D: \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} \) 满足 (D1) - (D4)

证

(i) 存在

(ii) 唯一：\( A \mapsto \) 相同 reduced echelon matrix \( \Rightarrow \det(A) \) 相同