Chapter 6 Eigenvalues and Eigenvectors

Eigenvalues & Matrix diagonization

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(特徵值, 固有值)

- **定義 1** $\begin{cases} T: \mathbb{R}^n \longrightarrow \mathbb{R}^n, \\ A = [T]_\varepsilon \end{cases} \exists \begin{cases} \lambda \in \mathbb{R} & T(v) = \lambda v, \\ v \neq 0 \in \mathbb{R}^n & Av = \lambda v. \end{cases}$ λ: eigen value of T, v: eigen vector of A

- **定理 1** $\exists$ invertible P, $P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \Lambda$ (A可对角化)

  $\Leftrightarrow \exists$ (eigen) basis $\mathcal{B} = \{v_1, \cdots, v_n\}$, $Av_i = \lambda_i v_i$ $(1 \le i \le n)$, $P = [v_1 \cdots v_n]$

  証 "$\Leftarrow$" 座標變換

  "$\Rightarrow$" $AP = [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n] = P\Lambda$

- **定理 2** (1) Eigen space $E(\lambda) := \{v \in \mathbb{R}^n \mid (A - \lambda I)v = 0\} = N(A - \lambda I) \triangleleft \mathbb{R}^n$

  (2) λ: eigenvalue of A $\Leftrightarrow E(\lambda) \neq \{0\} \Leftrightarrow A - \lambda I$ singular $\Leftrightarrow \det(A - \lambda I) = 0$

- **定義 2** $P_A(t) := \det(A - tI)$ (characteristic polynomial of A)

  例: (i) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $P_A(t) = \det\left(\begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}\right) = t^2 - \underline{(a+d)}t + \underline{(ad-bc)}$

  (ii) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $P_A(t) = \det\left(\begin{bmatrix} a_{11}-t & a_{12} & a_{13} \\ a_{21} & a_{22}-t & a_{23} \\ a_{31} & a_{32} & a_{33}-t \end{bmatrix}\right) = -\left[t^3 - \underline{tr(A)}t^2 + (\cdot\cdot)t - \underline{\det(A)}\right]$

- **定義 3** A similar to B $(A \sim B)$, if $\exists$ invertible P, $P^{-1}AP = B$

- **定理 3** $A \sim B \Rightarrow$ (1) $P_A(t) = P_B(t)$ $\Rightarrow$ A, B 有相同的特徵根

  (2) $tr(A) = tr(B)$

  (3) $\det(A) = \det(B)$

  (4) $rank(A) = rank(B)$

  証 $\det(B - tI) = \det(P^{-1}(A - tI)P)$
  $= \det(P^{-1})\det(A - tI)\det(P)$
  $= \det(A - tI)$

- **定理 4** (Cayley-Hamilton) $P_A(A) = 0$

  証 (by using Jordan form)

  例: $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, $P_A(t) = t^2 - 5t + 4$

  $\Rightarrow P_A(A) = A^2 - 5A + 4I = 0$

- **Remarks**

  (1) $P^{-1}AP = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Rightarrow \begin{cases} (1) \ A = P\Lambda P^{-1} \\ (2) \ A^k = P\Lambda^k P^{-1} = P \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} P^{-1} \end{cases}$

  (2) $x_k = Ax_{k-1} \Rightarrow x_k = A^k x_0 = P\Lambda^k \underline{P^{-1}x_0} = [v_1 \cdots v_n] \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

  $= \underline{\underline{c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n}}$

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A 可对角化?: $\forall \lambda$, m 重根, 可找到 m 個 eigen vectors.

Example 1 $A = \begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix}$

解 $P_A(t) = \det(A-tI) = \det\begin{bmatrix} 3-t & 1 \\ -3 & 7-t \end{bmatrix} = t^2-10t+24$, $\begin{cases} \lambda_1 = 4 \\ \lambda_2 = 6 \end{cases}$

$\begin{cases} (1)\ E(4): N(A-4I) = N\left(\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\right) & v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ (2)\ E(6): N(A-6I) = N\left(\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}\right) & v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{cases}$ $\begin{cases} P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \\ P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \end{cases}$

Example 2 $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

解 $P_A(t) = \det(A-tI) = \det\left(\begin{bmatrix} 1-t & 2 & 1 \\ 0 & 1-t & 0 \\ 1 & 3 & 1-t \end{bmatrix}\right) = -t(t-1)(t-2)$ $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{cases}$

(1) $E(0): N(A-0I) = N\left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(2) $E(1): N(A-I) = N\left(\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\right)$, $v_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

(3) $E(2): N(A-2I) = N\left(\begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 3 & -1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}$

Example 3 $A = \begin{bmatrix} -1 & 4 & 2 \\ -1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 & 1 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ algebraic geometric multiplicity

解 (A) $\det(A-tI) = -(t-1)^2(t-2)$ $\begin{cases} \lambda_1 = 1 & m_1 = 2 & d_1 = \dim E(1) = 2 \\ \lambda_2 = 2 & m_2 = 1 & d_2 = \dim E(2) = 1 \end{cases}$

(1) $E(1): N(A-I) = N\left(\begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix}\right)$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y+z \\ y \\ z \end{bmatrix}$, $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(2) $E(2): N(A-2I) = N\left(\begin{bmatrix} -3 & 4 & 2 \\ -1 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right)$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ z \\ z \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$

(B) $\det(B-tI) = -(t-1)^2(t-2)$ $\begin{cases} \lambda_1 = 1, & m_1 = 2, & d_1 = \dim E(1) = 1 \\ \lambda_2 = 2, & m_2 = 1, & \end{cases}$

(1) $E(1): N(B-I) = N\left(\begin{bmatrix} -1 & 3 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

不能對角化! (需2個 eigen vectors)

Example 4 (不能対角化)

(1) $A_{\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad P_A(t) = t^2+1$ $\quad$ (無實根) $\quad A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ $\quad P_A(t) = (t-2)^2$ $\quad E(2) = N\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)$, $\begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(3) $A = \left[\begin{array}{cc|cc} 2 & 0 & & \\ 0 & 2 & & \\ \hline & & 3 & 1 \\ & & 0 & 3 \end{array}\right]$ $\quad P_A(t) = (t-2)^2(t-3)^2$

(4) $B = \left[\begin{array}{cc|cc} 2 & 1 & & \\ 0 & 2 & & \\ \hline & & 3 & 0 \\ & & 0 & 3 \end{array}\right]$ $\quad P_A(t) = (t-2)^2(t-3)^2$

$$\begin{cases} E(2) = N\left(\left[\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & & \\ \hline & & 1 & 1 \\ & & 0 & 1 \end{array}\right]\right), \quad \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} & m_1 = 2,\ d_1 = 2 \\ E(3) = N\left(\left[\begin{array}{cc|cc} -1 & 0 & & \\ 0 & -1 & & \\ \hline & & 0 & 1 \\ & & 0 & 0 \end{array}\right]\right), \quad \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \\ 0 \end{bmatrix} & m_2 = 2,\ d_2 = 1 \end{cases}$$

結論: A 可対角化 $\iff \exists$ eigen basis $\{v_1, \cdots, v_n\}$

定理4 $\lambda_1, \cdots, \lambda_k$ distinct $\Rightarrow \{v_1, \cdots, v_k\}$ 獨立 $(0 \neq v_i \in E(\lambda_i))$

証 $\{v_1\} \longrightarrow \{v_1, v_2\} \xrightarrow{*} \{v_1, v_2, v_3\} \to \cdots \to \{v_1, \cdots, v_k\}$ 獨立

* 若 $v_3 = c_1v_1 + c_2v_2 \Rightarrow \lambda_3 v_3 = c_1\lambda_1 v_1 + c_2\lambda_2 v_2 \quad (= Av_3)$

$-)\ \lambda_3 v_3 = c_1\lambda_3 v_1 + c_2\lambda_3 v_2$

$0 = c_1(\lambda_1 - \lambda_3)v_1 + c_2(\lambda_2 - \lambda_3)v_2 \quad (*)$

推論 $v_1 + \cdots + v_k = 0 \Rightarrow v_i = 0,\ 1 \le i \le k.$
$v_i \in E(\lambda_i)$

定理5 $P_A(t) = \pm(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$, $d_i = \dim E(\lambda_i) \Rightarrow 1 \le d_i \le m_i \quad (1 \le i \le k)$

証 $\underbrace{\{v_1, \cdots, v_{d_i}\}}_{E(\lambda_i)\text{基底}} \cup \{v_{d_i+1}, \cdots, v_n\}$ 為 $\mathbb{R}^n$ 基底 $\quad A \sim \left[\begin{array}{c|c} \underbrace{\begin{matrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{matrix}}_{d_i} & B \\ \hline 0 & C \end{array}\right]$, $P_A(t) = (\lambda_i - t)^{d_i}\det(C - tI)$

$\Rightarrow d_i \le m_i$

定理6 (1) A 可対角化 (有 n 個獨立 eigen vectors) $\iff d_i = m_i \quad (1 \le i \le k) \quad (m_1 + \cdots + m_k = n)$

(2) n 個 eigen value $\Rightarrow A$ 可対角化.

証 (1) "$\Leftarrow$" $\{v_{i1}, \cdots, v_{im_i}\}$ $E(\lambda_i)$基底, $\sum_{i=1}^{k} \underbrace{\sum_{j=1}^{m_i} c_{ij}v_{ij}}_{v_i} = 0 \xrightarrow[\text{推論}]{\text{定理4}} \sum_{j=1}^{m_i} c_{ij}v_{ij} = 0 \Rightarrow c_{ij} = 0 \begin{pmatrix} 1 \le i \le k \\ 1 \le j \le m_i \end{pmatrix} \Rightarrow \{v_{ij}\}$ 獨立 (基底)

"$\Rightarrow$"

$$A \sim \begin{bmatrix} \begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix} & & & \\ & \begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} \lambda_k & & \\ & \ddots & \\ & & \lambda_k \end{matrix} \end{bmatrix} \begin{matrix} \le d_1 \\ \\ \le d_2 \\ \\ \underbrace{\le d_k} \end{matrix}$$

$n \le d_1 + d_2 + \cdots + d_k$

$\le m_1 + m_2 + \cdots + m_k = n$

$\Rightarrow d_i = m_i$

$(\because \dim E(\lambda_k) = d_k)$

例 $P_A(t) = (t-2)(t-4)^3(t-5)^6$, $\begin{cases} \lambda_1 = 2, & \lambda_2 = 4, & \lambda_3 = 5 \\ m_1 = 1, & m_2 = 3, & m_3 = 6 \end{cases}$

Applications

$$A: \begin{cases} \lambda_1, & v_1 \\ \vdots & \vdots \\ \lambda_n, & v_n \end{cases}, \quad P = [v_1 \cdots v_n], \quad \begin{cases} x = Py \\ A = P\Lambda P^{-1} \end{cases}$$

(A) Difference equation system: $x_k = A x_{k-1}, \ x_0$

$$x_k = A x_{k-1} = A^k x_0 = P \Lambda^k \underline{P^{-1} x_0} = [v_1 \cdots v_n] \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \underline{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}} = \underline{c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n}$$

- Example 1 (生態系平衡: cat/mouse problem) (p.69, 297)

$$\begin{cases} (\text{cat population at month } k) & c_k = 0.7 c_{k-1} + 0.2 m_{k-1}, \\ (\text{mouse} \quad '' \quad\quad '' \quad) & m_k = -0.6 c_{k-1} + 1.4 m_{k-1}, \end{cases} \quad \underbrace{\begin{bmatrix} c_k \\ m_k \end{bmatrix}}_{x_k} = \underbrace{\begin{bmatrix} 0.7 & 0.2 \\ -0.6 & 1.4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_{k-1} \\ m_{k-1} \end{bmatrix}}_{x_{k-1}}, \quad \underbrace{\begin{bmatrix} c_0 \\ m_0 \end{bmatrix}}_{x_0}$$

$$A = \begin{bmatrix} 0.7 & 0.2 \\ -0.6 & 1.4 \end{bmatrix} \begin{cases} \lambda_1 = 1, & v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ \lambda_2 = 1.1, & v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{cases}, \quad P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad P^{-1} x_0 = \begin{bmatrix} 2c_0 - m_0 \\ 2m_0 - 3c_0 \end{bmatrix}, \quad \underline{x_k = (2c_0 - m_0) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (2m_0 - 3c_0)(1.1)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\begin{cases} \text{(i)} \ 2m_0 = 3c_0 & x_k \equiv (2c_0 - m_0) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ \text{(ii)} \ 2m_0 > 3c_0 & x_k \to \begin{bmatrix} \infty \\ \infty \end{bmatrix}, \quad m_k \approx 2c_k \quad (\text{birth rate double}) \\ \text{(iii)} \ 2m_0 < 3c_0 & x_k \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{mouse disappear first} \ (\text{death} \ '' \ '') \end{cases}$$

(B) Differential equation system: $x' = Ax, \ x(0)$, $\quad x = Py \Rightarrow \begin{cases} x' = Py' \\ x(0) = Py(0) \end{cases}$

$$\begin{cases} Py' = APy \\ y' = \Lambda y \end{cases} \quad \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \end{bmatrix}, \quad \underline{\begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}}, \quad \underline{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}} = y(0) = \underline{P^{-1} x(0)}$$

$$\underline{x(t)} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = [v_1 \cdots v_n] \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} = \underline{c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n} \qquad \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} \right)$$

$$\left(y' = \lambda y \Rightarrow y = c e^{\lambda t} \right)$$

- Example 2

$$\begin{cases} x'(t) = 3x(t) + y(t), & x(0) = 1, \\ y'(t) = 2x(t) + 2y(t), & y(0) = 4. \end{cases} \quad \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{x'} = \underbrace{\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x}, \quad x(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}, \begin{cases} \lambda_1 = 4, & v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda_2 = 1, & v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{cases} \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad P^{-1} x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \underline{2e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 2e^{4t} - e^{t} \\ 2e^{4t} + 2e^{t} \end{bmatrix}$$

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Example 3 (Fibonacii) 解 $a_{k+1} = a_{k-1} + a_k$, $k \geq 1$, $a_0 = a_1 = 1$

解: $\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{k-1} \\ a_k \end{bmatrix}$, $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \left(A\mathbb{v}_1 = \begin{bmatrix} \lambda_1 \\ 1+\lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1^2 \end{bmatrix} = \lambda_1 \mathbb{v}_1 \right)$

$$\mathbb{x}_k = A\mathbb{x}_{k-1},\quad \mathbb{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ P_A(t) = t^2 - t - 1,\ \begin{cases} \lambda_1 = \frac{1-\sqrt{5}}{2}, & \mathbb{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \\ \lambda_2 = \frac{1+\sqrt{5}}{2} & \mathbb{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \end{cases}$$

$$= P\Lambda P^{-1}$$

$$\begin{aligned} \mathbb{x}_k &= A^k \mathbb{x}_0 \\ &= P\Lambda^k P^{-1}\mathbb{x}_0 \\ &= \underline{\frac{-\lambda_1}{\sqrt{5}} \lambda_1^k \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \frac{\lambda_2}{\sqrt{5}} \lambda_2^k \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}} \end{aligned}$$

$$\begin{cases} P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} & (\lambda_1 + \lambda_2 = 1) \\ P^{-1}\mathbb{x}_0 = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_1 \\ \lambda_2 \end{bmatrix} \end{cases}$$

$$\therefore a_k = \underline{\frac{1}{\sqrt{5}} \left(\lambda_2^{k+1} - \lambda_1^{k+1} \right)} \sim \frac{1}{\sqrt{5}} \lambda_2^{k+1} \quad \left(\because \left| \frac{\lambda_1}{\lambda_2} \right| < 1 \right)$$

(Cribbage Match)

Example 4

$$\begin{cases} p_k = P_r\{A \text{ wins at time } k\}, \\ q_k = P_r\{B \text{ wins } \quad " \quad \}, \end{cases} \begin{cases} p_{k+1} = 0.6p_k + 0.45q_k \\ q_{k+1} = 0.4p_k + 0.55q_k \end{cases} \Rightarrow \begin{bmatrix} p_{k+1} \\ q_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.6 & 0.45 \\ 0.4 & 0.55 \end{bmatrix}}_{A \text{ (轉移矩陣)}} \begin{bmatrix} p_k \\ q_k \end{bmatrix}$$

$\mathbb{x}_k = \begin{bmatrix} p_k \\ q_k \end{bmatrix}$ $(p_k + q_k = 1)$, $\quad \underline{\mathbb{x}_{k+1} = A\mathbb{x}_k}$

觀察:

(甲) (p72)

$$\begin{cases} \mathbb{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbb{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \mathbb{x}_3 = \begin{bmatrix} 0.54 \\ 0.46 \end{bmatrix}, \ldots, \mathbb{x}_{100} = \begin{bmatrix} 0.52941 \\ 0.47059 \end{bmatrix} \\ \mathbb{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbb{x}_2 = \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix}, \mathbb{x}_3 = \begin{bmatrix} 0.5175 \\ 0.4825 \end{bmatrix}, \ldots, \mathbb{x}_{100} = \begin{bmatrix} 0.52941 \\ 0.47059 \end{bmatrix} \sim \frac{1}{17} \begin{bmatrix} 9 \\ 8 \end{bmatrix} = \mathbb{x}_\infty \end{cases}$$

(乙) $\mathbb{x}_\infty = \lim_{k\to\infty} \mathbb{x}_k$, $\Rightarrow A\mathbb{x}_\infty = \mathbb{x}_\infty$. $\quad (\mathbb{x}_\infty \in E(1))$

解: (p280)

$$A = \begin{bmatrix} 0.6 & 0.45 \\ 0.4 & 0.55 \end{bmatrix},\ P_A(t) = t^2 - 1.15t + 0.15,\ \begin{cases} \lambda_1 = 1, & \mathbb{v}_1 = \begin{bmatrix} 9 \\ 8 \end{bmatrix}, \\ \lambda_2 = 0.15, & \mathbb{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases} \quad 1 = \lambda_1 > \lambda_2 = 0.15$$

$$\begin{aligned} \mathbb{x}_k &= A\mathbb{x}_{k-1} \\ &= A^{k-1}\mathbb{x}_1 \\ &= P\Lambda^{k-1} P^{-1}\mathbb{x}_1 \\ &= c_1 \lambda_1^{k-1} \mathbb{v}_1 + c_2 \lambda_2^{k-1} \mathbb{v}_2 \\ &= \underline{\frac{1}{17} \begin{bmatrix} 9 \\ 8 \end{bmatrix} + c_2 (0.15)^{k-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \end{aligned}$$

$$P = \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix},\ P^{-1}\mathbb{x}_1 = \frac{1}{17} \begin{bmatrix} 1 & 1 \\ -8 & 9 \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 1 \\ * \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\xrightarrow{k\to\infty} \frac{1}{17} \begin{bmatrix} 9 \\ 8 \end{bmatrix} = \mathbb{x}_\infty \in E(1) = \operatorname{span}(\mathbb{v}_1)$$

(indep of $\mathbb{x}_1$)

$$\Rightarrow A\mathbb{x}_\infty = \mathbb{x}_\infty$$

Markov chain

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- $\mathbb{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$: probability vector, if $1 \le i \le n$, $x_i \ge 0$, $\sum_{i=1}^{n} x_i = 1$.
- $A = [a_1 \cdots a_n]$ stochastic matrix, if 〃 a_i is probability vector. ($A\mathbb{x}$ is probability vector)
- A regular 〃 , if $\exists k$, A^k 的所有元素為正.

Lemma 3.1 A: stochastic

(1) 1 is eigen value of A^T

(2) 1 is eigen value of A ($\mu \ne 1$ 特徵根 $\Rightarrow |\mu| < 1$)

證 $A^T\mathbb{x} = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$\det(A-I) = \det(A-I)^T = \det(A^T - I) = 0$

Proposition 3.2 A: regular stochastic $\Rightarrow$ dim E(1) = 1 (令 $\mathbb{v}$ = probability vector $\in$ E(1))

定理 3.3 $A_{n\times n}$: regular stochastic $(n>1)$

(1) $\lim_{k\to\infty} A^k = [\mathbb{v}\ \mathbb{v} \cdots \mathbb{v}]$

(2) $\lim_{k\to\infty} A^k \mathbb{x}_0 = \mathbb{v}$, $\forall$ probability vector $\mathbb{x}_0$ ($\mathbb{x}_k \to \mathbb{v}$)

定義

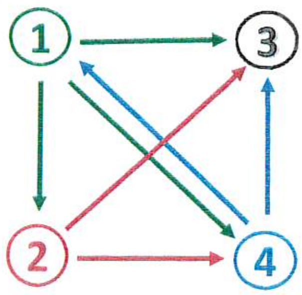
$G = (V, E)$, $V = \{1, 2, \ldots, n\}$ states, $(i \to j) \Leftrightarrow (i,j) \in E \subset V \times V$

(1) $X_k :=$ state at time k, $k \ge 0$, state vector $\mathbb{x}_k = \begin{bmatrix} Pr(X_k = 1) \\ \vdots \\ Pr(X_k = n) \end{bmatrix}$ is probability vector

(2) transition matrix: $A = [a_{ij}]_{n\times n}$, $a_{ij} = Pr(X_k = i \mid X_{k-1} = j)$ $\Rightarrow$ A is stochastic

(3) Markov chain: $\{\mathbb{x}_k\}_{k\ge 0}$, $\mathbb{x}_k = A\mathbb{x}_{k-1}$ $(= A^k \mathbb{x}_0)$

Example (Pagerank): $a_{ij} = \begin{cases} p\,\dfrac{\delta_{(j,i)\in E}}{out(j)} + (1-p)\,\dfrac{1}{n}, & out(j) > 0 \\ \dfrac{1}{n}, & 〃 = 0 \end{cases}$



- $A = p\begin{bmatrix} 0 & 0 & 1/4 & 1/2 \\ 1/3 & 0 & 1/4 & 0 \\ 1/3 & 1/2 & 1/4 & 1/2 \\ 1/3 & 1/2 & 1/4 & 0 \end{bmatrix} + (1-p)\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

- $p = 1$, $\lambda = 1$, $x_{10} = x_{11} = \mathbb{v} = \frac{1}{97}\begin{bmatrix} 21 \\ 16 \\ 36 \\ 24 \end{bmatrix} = \begin{bmatrix} 0.2165 \\ 0.1649 \\ 0.3711 \\ 0.2474 \end{bmatrix}$

- $p = 0.85$, $\lambda = 1$, $x_9 = x_{10} = \mathbb{v} = \begin{bmatrix} 0.2192 \\ 0.1752 \\ 0.3558 \\ 0.2497 \end{bmatrix}$