Chapter 7 Symmetric Matrices & Singular Value Decompositions

Orthogonal matrices (正交矩陣)

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- **定義**: $Q_{n\times n}$ orthogonal : $\{q_1, q_2, \dots, q_n\}$ orthonormal

- **定理**: 下列 (1) – (5) 等價:

(1) Q orthogonal

(2) $Q^TQ = I_n$

(3) $Q\mathbf{x}\cdot Q\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$ (保積)

(4) $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ (保長)

(5) $\|Q\mathbf{x} - Q\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$ (保距)

証

(1) ⇒ (2) $\begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \end{bmatrix}\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) ⇒ (3) $Q\mathbf{x}\cdot Q\mathbf{y} = \mathbf{x}\cdot Q^TQ\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$

(3) ⇒ (1) $q_i\cdot q_j = Q\mathbf{e}_i\cdot Q\mathbf{e}_j = \mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$

(3) ⇒ (4) $\|Q\mathbf{x}\|^2 = Q\mathbf{x}\cdot Q\mathbf{x} = \mathbf{x}\cdot\mathbf{x} = \|\mathbf{x}\|^2$

(4) ⇒ (5) $\|Q\mathbf{x}-Q\mathbf{y}\| = \|Q(\mathbf{x}-\mathbf{y})\| = \|\mathbf{x}-\mathbf{y}\|$

(4) ⇐ (5) $\mathbf{y} = \mathbf{0}$

- **Remarks** $\left\{\begin{matrix} 保長 \\ 保積 \end{matrix}\right.$ ⇒ $\begin{matrix} 保角 \\ 保面積 \end{matrix}$ $\left(\cos\theta = \dfrac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right)$

(4,5) ⇒ (3) $\begin{cases} \|Q\mathbf{x}-Q\mathbf{y}\|^2 = \|Q\mathbf{x}\|^2 - 2\,Q\mathbf{x}\cdot Q\mathbf{y} + \|Q\mathbf{y}\|^2 \\ \|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\,\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \end{cases}$

- **定理** Q: orthogonal $(Q^TQ = I_n)$ 則

(1) $\det(Q) = \pm 1$ ∵ $\det(Q^T)\det(Q) = 1$

(2) $QQ^T = I_n$ ⇒ $\{Q_1, \dots, Q_n\}$ (Q的列) orthonormal

(3) $Q^{-1} = Q^T$ orthogonal $(Q^T)^TQ^T = I_n$

(4) P, Q orthogonal ⇒ PQ orthogonal $(PQ)^TPQ = Q^TP^TPQ = I_n$

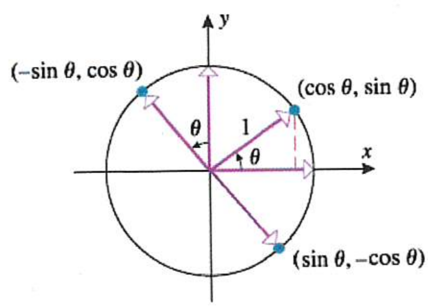
- **定理** $\begin{cases} O_n = \{A_{n\times n} \mid A^TA = I_n\} & \text{(general) orthogonal group} \\ SO_n = \{A_{n\times n} \mid A^TA = I_n,\ \det(A) = 1\} & \text{Special 〃 〃} \end{cases}$

- **定理** $Q \in O_2$

(1) $\det(Q) = 1$, $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R_\theta$: $\mathbf{x}_\varphi \to \mathbf{x}_{\theta+\varphi}$, $\mathbf{x}_\varphi = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$

(2) $\det(Q) = -1$, $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = H_\theta$: $\mathbf{x}_\varphi \to \mathbf{x}_{\theta-\varphi}$

(对 $L_{\frac{\theta}{2}}$ 作鏡射)



- **定理**

(1) $R_\alpha R_\beta = R_{\alpha+\beta}$ $\varphi \to \alpha + (\beta+\varphi)$

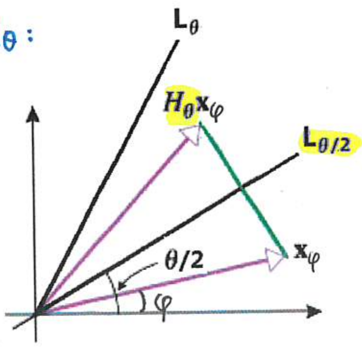
(2) $H_\alpha R_\beta = H_{\alpha-\beta}$ $\to \alpha - (\beta+\varphi)$

(3) $R_\alpha H_\beta = H_{\alpha+\beta}$ $\to \alpha + (\beta-\varphi)$

(4) $H_\alpha H_\beta = R_{\alpha-\beta}$ $\to \alpha - (\beta-\varphi)$

或 $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix}$

H_θ :



Symmetric Matrix & Spectral theorem

• **定理1** (Spectral th) $A_{n\times n}$: symmetric $(A^T = A)$

(1) $\begin{cases} A\mathbf{x} = \lambda \mathbf{x} \\ A\mathbf{y} = \mu \mathbf{y}\end{cases}$ $\lambda \neq \mu \Rightarrow \mathbf{x}\cdot\mathbf{y} = 0$

(2) all eigenvalues $\in \mathbb{R}$

(3) $\begin{cases} \exists \text{ orthonormal eigenbasis } \{q_1,\dots,q_n\} \Leftrightarrow \\ \exists \text{ orthogonal } Q=[q_1\cdots q_n],\ Q^TAQ = Q^{-1}AQ = \Lambda \end{cases}$

(4) $A = Q\Lambda Q^T$ $= [q_1\ q_2 \cdots q_n] \begin{bmatrix}\lambda_1 &&& \\ &\lambda_2&& \\ &&\ddots& \\ &&&\lambda_n\end{bmatrix}\begin{bmatrix}q_1^T\\q_2^T\\\vdots\\q_n^T\end{bmatrix}$

$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$ $= [\lambda_1 q_1\ \lambda_2 q_2 \cdots \lambda_n q_n]\begin{bmatrix}q_1^T\\q_2^T\\\vdots\\q_n^T\end{bmatrix}$

$= \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$

證 (1) $\lambda(\mathbf{x}\cdot\mathbf{y}) = A\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot A^T\mathbf{y} = \mathbf{x}\cdot A\mathbf{y} = \mu(\mathbf{x}\cdot\mathbf{y})$

(2) $\begin{cases}A\mathbf{x}=\lambda \mathbf{x}\\ \lambda\in\mathbb{C},\ \mathbf{x}\neq 0\end{cases} \Rightarrow \begin{cases} A\mathbf{x} = \lambda \mathbf{x} \\ A\bar{\mathbf{x}} = \bar\lambda \bar{\mathbf{x}}\end{cases} \xRightarrow[(\ )^T\mathbf{x}]{} \begin{cases}\bar{\mathbf{x}}^T A \mathbf{x} = \lambda \bar{\mathbf{x}}^T \mathbf{x} \\ \bar{\mathbf{x}}^T A^T \mathbf{x} = \bar\lambda \bar{\mathbf{x}}^T \mathbf{x}\end{cases} \xRightarrow{\mathbf{x}\neq 0} \lambda=\bar\lambda\in\mathbb{R}$

(3) $\begin{cases} 取\ q_1,\ \|q_1\|=1,\ Aq_1=\lambda_1 q_1 \\ \text{Span}(q_1)^\perp\ 的\ \text{orthonormal basis}\ \{v_2,\dots,v_n\} \end{cases} \Rightarrow \{q_1, v_2,\dots,v_n\}: \mathbb{R}^n\ 的\ \text{orthonormal basis}$

$\Rightarrow A\sim B = \left[\begin{array}{c|c}\lambda_1 & * \cdots * \\ \hline 0 & \\ \vdots & C \\ 0 & \end{array}\right] = Q^{-1}AQ \xRightarrow{B^T=B} \begin{cases} * = 0 \\ C: 对稱\end{cases} \xRightarrow{\text{induction}} \begin{cases}\exists\ \text{Span}(q_1)^\perp\ 的\ \text{orthonormal basis} \\ \{q_2,\dots,q_n\}\end{cases}$

$Cq_i = \lambda_i q_i$, $2\le i\le n$

$\Rightarrow \{q_1,\dots,q_n\}$ $\mathbb{R}^n$ orthonormal eigenbasis $\Rightarrow A\sim \begin{bmatrix}\lambda_1 &&& \\ &\lambda_2&& \\ &&\ddots& \\ &&&\lambda_n\end{bmatrix}$

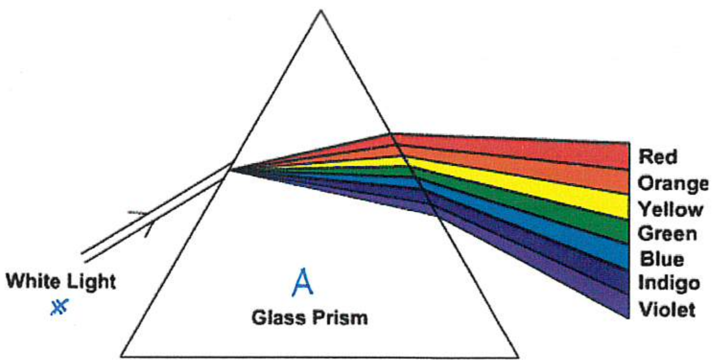
• **定理2** A: symmetric $\Leftrightarrow A = Q\Lambda Q^T,\ Q^TQ = I_n$

• **Remarks** (Spectral decomposition)

$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T,$ $\quad \mathbf{x} = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n,$ $\quad q_i^T q_j = q_i\cdot q_j$

$= \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$

$$\begin{array}{rcccc} & A: & \mathbf{x} & \longrightarrow & A\mathbf{x}\\ & \| & \| & & \| \\ \lambda_1 q_1 q_1^T = & \lambda_1 P_1: & c_1 q_1 & \longrightarrow & \lambda_1 c_1 q_1 \\ & + & + & & + \\ & \lambda_2 P_2: & c_2 q_2 & \longrightarrow & \lambda_2 c_2 q_2 \\ & + & + & & + \\ & \vdots & \vdots & & \vdots \\ & + & + & & + \\ & \lambda_n P_n: & c_n q_n & \longrightarrow & \lambda_n c_n q_n \end{array}$$



$\begin{cases} A: 稜鏡 \\ \mathbf{x}: 白光 \end{cases}$ $\begin{cases} q_1,\dots,q_n & 有色光 \\ \lambda_1,\dots,\lambda_n & 折射率 \end{cases}$

Example 1

$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$, $P_A(t) = t^2 + t - 6 = 0$

$$\begin{cases} \lambda_1 = 2, & v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, & q_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \lambda_2 = -3, & v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, & q_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{cases} \qquad Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

(i) $Q^T A Q = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

(ii) $A = Q \Lambda Q^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$

$$= \frac{2}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix} + \frac{-3}{5}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \end{bmatrix}$$

$$= \frac{2}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} + \frac{-3}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = 2P_1 - 3P_2$$

註 $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $b \neq 0 \Rightarrow \lambda_1 \neq \lambda_2$ $\quad (\Delta = (a-c)^2 + b^2 \geq 0)$

$P_A(t) = t^2 - (a+c)t + (ac - b^2) = 0$

$$\begin{cases} \lambda_1, & v_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \\ \lambda_2, & v_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix} \end{cases} \qquad v_1 \cdot v_2 = 0$$

$$A v_1 = \begin{bmatrix} b\lambda_1 \\ b^2 + c\lambda_1 - ac \end{bmatrix} = \begin{bmatrix} b\lambda_1 \\ \lambda_1^2 - a\lambda_1 \end{bmatrix} = \lambda_1 v_1$$

Example 2

$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $P_A(t) = -t^3 + 2t^2 + t - 2 = -(t+1)(t-1)(t-2)$

$$\begin{cases} \lambda_1 = -1, & v_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, & q_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\ \lambda_2 = 1, & v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, & q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ \lambda_3 = 2, & v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, & q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{cases}$$

$$Q = \begin{bmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \qquad Q^T A Q = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

Example 3

$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$, $P_A(t) = -t^3 + 18t^2 - 81t = -t(t-9)^2$ (重根沒問題)

$\lambda_1 = 0$, $v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ $\qquad q_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 9$, $A - 9I = \begin{bmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ -2 & -2 & -1 \end{bmatrix}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y - \frac{1}{2}z \\ y \\ z \end{bmatrix}$

$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $\qquad q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$w_3 = v_3 - \frac{v_3 \cdot v_2}{\|v_2\|^2} v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{bmatrix} \parallel \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \qquad q_3 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{2}{3} & \frac{-1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}, \qquad Q^T A Q = \begin{bmatrix} 0 & & \\ & 9 & \\ & & 9 \end{bmatrix}$$

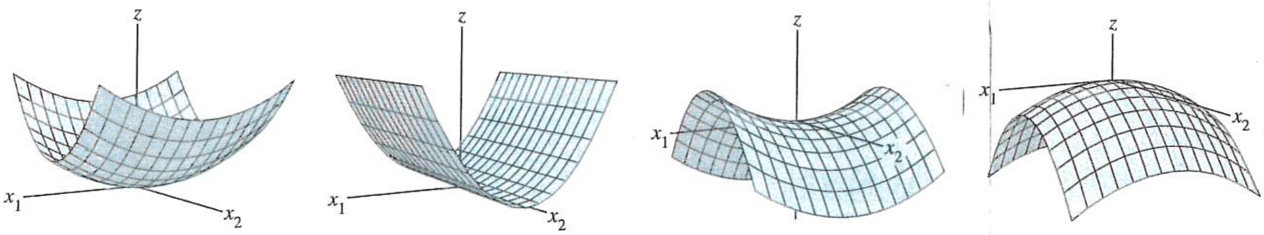
$A_{n\times n}$ 对稱 $\Rightarrow \exists\, Q=[q_1 \cdots q_n]$, $Q^TAQ = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$, $\{q_1,\dots,q_n\}$ orthonormal, $Aq_i=\lambda_i q_i$

- **定理1** (Principal axes) $\quad F(\mathbf{x}) = \mathbf{x}^TA\mathbf{x} \xlongequal{\mathbf{x}=Q\mathbf{y}} \mathbf{y}^T\Lambda\mathbf{y}$ $\quad$: q_i 主軸

$$F(x_1,\dots,x_n) = \sum_{i=1}^{n} a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_ix_j = \lambda_1 y_1^2+\cdots+\lambda_n y_n^2 \quad \text{(新座標系)}$$

- **定理2** $\quad$ **Quadratic form:** $\quad F(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$

$F(\mathbf{x})$ 或 A 為：	當	判別法： 所有特徵值 λ_i
正定 (positive definite)	$F(\mathbf{x}) > 0, \quad \forall\, \mathbf{x}\neq \mathbf{0}$	全部為正
負定 (negtive definite)	$F(\mathbf{x}) < 0, \quad \forall\, \mathbf{x}\neq \mathbf{0}$	全部為負
不定 (indefinite)	$F(x)$ 有正有負, $\forall\, \mathbf{x}\neq \mathbf{0}$	有正有負
半正定 (semi-positive definite)	$F(\mathbf{x}) \geq 0, \quad \forall\, \mathbf{x}\in \mathbf{R}^n$	全部 ≥ 0
半負定 (semi-negtive definite)	$F(\mathbf{x}) \leq 0, \quad \forall\, \mathbf{x}\in \mathbf{R}^n$	全部 ≤ 0



(a) $z=3x_1^2+7x_2^2$ 正定 $\quad$ (b) $z=3x_1^2$ 半正定 $\quad$ (c) $z=3x_1^2-7x_2^2$ 不定 $\quad$ (d) $z=-3x_1^2-7x_2^2$ 負定

- **定理3** (Constraint optimization)

$F(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$, $\|\mathbf{x}\|=1$, 則 $\begin{cases} M = \lambda_1 \text{ at } q_1 \\ m = \lambda_n \text{ 〃 } q_n \end{cases}$

$(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$

- **定理4** $\quad F(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$, 則 $\max\{F(\mathbf{x}) \mid \|\mathbf{x}\|=1,\ \mathbf{x}^Tq_1=0\} = \lambda_2$ at q_2

$$\mathbf{x}^TA\mathbf{x} \xlongequal{\mathbf{x}=Q\mathbf{y}} \mathbf{y}^T\Lambda\mathbf{y} = \lambda_1y_1^2+\lambda_2y_2^2+\cdots+\lambda_ny_n^2 = \lambda_2y_2^2+\cdots+\lambda_ny_n^2 \qquad M=\lambda_2 \text{ at } e_2$$

$\|\mathbf{x}\|=1 \qquad \|\mathbf{y}\|=1 \qquad y_1^2+y_2^2+\cdots+y_n^2=1 \qquad y_2^2+\cdots+y_n^2=1 \qquad q_2$

$\mathbf{x}^Tq_1=0 \qquad \mathbf{y}^Te_1=0 \qquad [y_1 \cdots y_n]\begin{bmatrix}1\\0\\ \vdots \\0\end{bmatrix} = y_1 = 0$

- **定理5** $\quad F(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$, 則 $\max\{F(\mathbf{x}) \mid \|\mathbf{x}\|=1,\ \mathbf{x}^Tq_1=\mathbf{x}^Tq_2=\cdots=\mathbf{x}^Tq_{k-1}=0\} = \lambda_k$ at q_k

- **定理6** (Principal component analysis)

$X_{d\times n}$: data matrix, $\frac{1}{n}XX^T$: eigen values $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$

$\Rightarrow \begin{cases} \text{1st PC at } q_1 \\ \text{2nd 〃 〃 } q_2 \\ \quad\vdots \\ d\text{th 〃 〃 } q_d \end{cases}$

Quadratic Form（二次式）

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$$\begin{cases} F(x,y) = [x\ y]\begin{bmatrix} a & b \\ b & c\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = ax^2 + \underbrace{2bxy}_{\text{消除}} + cy^2 \\ F(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \mathbf{y}^T Q^T A Q \mathbf{y} \\ \quad = \mathbf{y}^T \Lambda \mathbf{y} = \lambda_1 x'^2 + \lambda_2 y'^2 \end{cases}$$

A: 对稱

$$\begin{cases} \mathbf{x} = Q\mathbf{y} \\ Q^T A Q = \Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \end{cases}$$

Example 1 化簡二次曲線 $x^2+4xy-2y^2=6$

$$[x\ y]\begin{bmatrix} 1 & 2 \\ 2 & -2\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = 6, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & -2\end{bmatrix}, \quad Q = \frac{1}{\sqrt5}\begin{bmatrix} 2 & -1 \\ 1 & 2\end{bmatrix}$$

q_1, q_2

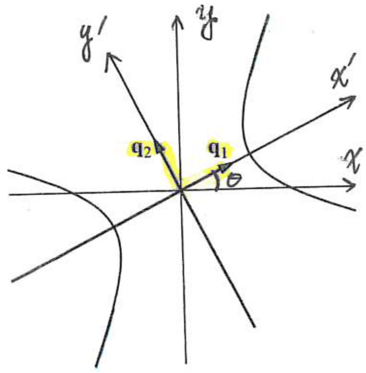
$$\mathbf{x}^T A \mathbf{x} = 6 \qquad \begin{cases} Q^TAQ = \begin{bmatrix} 2 & 0 \\ 0 & -3\end{bmatrix} = \Lambda \\ \mathbf{x} = Q\mathbf{y}\end{cases}$$

$$\mathbf{y}^T Q^T A Q \mathbf{y} = 6$$

$$\mathbf{y}^T \Lambda \mathbf{y} = 6$$

$$[x'\ y']\begin{bmatrix} 2 & 0 \\ 0 & -3\end{bmatrix}\begin{bmatrix} x' \\ y'\end{bmatrix} = 6$$

$$2x'^2 - 3y'^2 = 6 \quad \text{双曲線}$$



$$\begin{cases} \cot 2\theta = \frac{a-c}{2b} = \frac{3}{4} \ (\text{高中}) \\ \tan\theta = \frac{1}{2} \Rightarrow q_1 = \frac{1}{\sqrt5}\begin{bmatrix} 2 \\ 1\end{bmatrix} \\ \tan 2\theta = \frac{2\cdot\frac12}{1-\frac14} = \frac{4}{3} \end{cases}$$

Example 2 $x^2+4xy-2y^2+2x+4y = 6$

$$[x\ y]\begin{bmatrix} 1 & 2 \\ 2 & -2\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} + [2\ 4]\begin{bmatrix} x \\ y\end{bmatrix} = 6$$

$$\mathbf{x}^T A \mathbf{x} + [2\ 4]\mathbf{x} = 6$$

$$\mathbf{y}^T Q^T A Q \mathbf{y} + [2\ 4]Q\mathbf{y} = 6$$

$$[x'\ y']\begin{bmatrix} 2 & 0 \\ 0 & -3\end{bmatrix}\begin{bmatrix} x' \\ y'\end{bmatrix} + [2\ 4]\frac{1}{\sqrt5}\begin{bmatrix} 2 & -1 \\ 1 & 2\end{bmatrix}\begin{bmatrix} x' \\ y'\end{bmatrix} = 6$$

$$2x'^2 - 3y'^2 + \frac{8}{\sqrt5}x' + \frac{6}{\sqrt5}y' = 6$$

$$2\left(x'+\frac{2}{\sqrt5}\right)^2 - 3\left(y'-\frac{1}{\sqrt5}\right)^2 = 7 \quad \text{双曲線, 中心}\left(\frac{-2}{\sqrt5}, \frac{1}{\sqrt5}\right)$$

$$\begin{cases} \lambda_1, \lambda_2 > 0 & \text{橢圓} \\ \quad < 0 & \text{双曲線} \\ \quad = 0 & \text{拋物線} \end{cases}$$

Example 3 $5x^2+5y^2+8z^2-8xy-4yz-4zx+2x+2y+z = 9$

$$[x\ y\ z]\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8\end{bmatrix}\begin{bmatrix} x \\ y \\ z\end{bmatrix} + [2\ 2\ 1]\begin{bmatrix} x \\ y \\ z\end{bmatrix} = 9$$

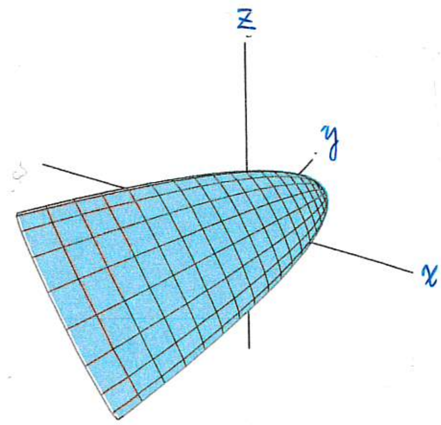
$$\mathbf{x}^T A \mathbf{x} + [2\ 2\ 1]\mathbf{x} = 9$$

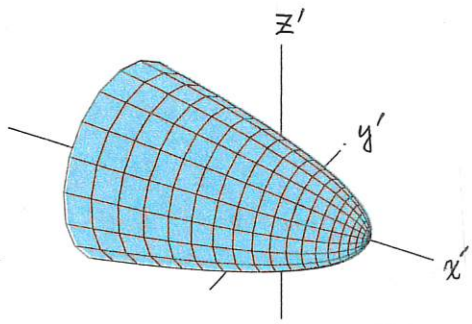
$$\mathbf{y}^T Q^T A Q \mathbf{y} + [2\ 2\ 1]Q\mathbf{y} = 9$$

$$[x'\ y'\ z']\begin{bmatrix} 0 & & \\ & 9 & \\ & & 9\end{bmatrix}\begin{bmatrix} x' \\ y' \\ z'\end{bmatrix} + [2\ 2\ 1]\begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z'\end{bmatrix} = 9$$

$$9y'^2 + 9z'^2 + 3x' = 9$$

$$x' = 3(1 - y'^2 - z'^2) \quad (\text{Paraboloid})$$



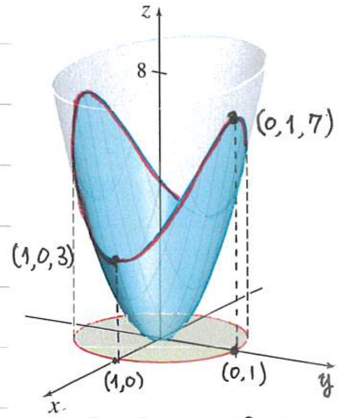


Constraint Optimization

求 $\begin{cases} F(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \\ F(x,y) = [x\ y]\begin{bmatrix} a & b \\ b & c\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = ax^2+2bxy+cy^2 \end{cases}$ 在 $\begin{cases} \|\mathbf{x}\| = 1 \\ x^2+y^2 = 1 \end{cases}$ 之 極大值 M 極小值 m

- **Example 1** $F(x,y) = 3x^2+7y^2, \quad x^2+y^2=1$ $\begin{cases} M = 7 \\ m = 3 \end{cases}$ at $\begin{matrix} \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{matrix}$

$$3 = 3x^2+3y^2 \le 3x^2+7y^2 \le 7x^2+7y^2 = 7$$



$z = 3x^2+7y^2 \cap x^2+y^2=1$

- **Example 2** $F(x,y) = 2x^2-3y^2, \quad x^2+y^2=1$ $\begin{cases} M = 2 \\ m = -3 \end{cases}$ at $\begin{matrix} \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix}$

$$-3 = -3x^2-3y^2 \le 2x^2-3y^2 \le 2x^2+2y^2 = 2$$

- **Example 3** $F(x,y) = x^2+4xy-2y^2, \quad x^2+y^2=1$ $\begin{cases} M = \lambda_1 = 2 \\ m = \lambda_2 = -3 \end{cases}$ at $\begin{matrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{matrix}$

$$\begin{cases} F(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = x^2+4xy-2y^2, \quad x^2+y^2=1, \quad \|\mathbf{x}\|=1 \quad \{\mathbf{q}_1, \mathbf{q}_2\}, \quad \{\mathbf{x}^T A\mathbf{x} \mid \|\mathbf{x}\|=1\} \\ \quad \| \quad \mathbf{x} = Q\mathbf{y}, \qquad \text{1-1, onto} \uparrow \qquad \| \\ F(\mathbf{y}) = \mathbf{y}^T \Lambda \mathbf{y} = 2u^2-3v^2, \quad u^2+v^2=1, \quad \|\mathbf{y}\|=1, \quad \{\mathbf{e}_1, \mathbf{e}_2\}, \quad \{\mathbf{y}^T\Lambda\mathbf{y} \mid \|\mathbf{y}\|=1\} \end{cases}$$

$$M = 2, \quad m = -3$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{cases} \lambda_1 = 2 \\ \lambda_2 = -3 \end{cases} \quad Q = \frac{1}{\sqrt{5}}\begin{array}{c} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\ \mathbf{q}_1 \quad \mathbf{q}_2 \end{array}, \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

- **Example 4** $F(x,y) = 2x^2+12xy-9y^2, \quad 4x^2+9y^2=36, \quad \begin{cases} x = 3u \\ y = 2v \end{cases} \quad \left(\frac{x^2}{9}+\frac{y^2}{4}=1\right)$

$$= 18(u^2+4uv-2v^2), \quad u^2+v^2=1,$$

$$\begin{cases} M = 2\cdot 18 = 36 \\ m = -3\cdot 18 = -54 \end{cases} \text{ at } \begin{matrix} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{matrix} \text{ or } \begin{matrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} 6 \\ 2 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} -3 \\ 4 \end{bmatrix} \end{matrix}$$

- **Example 5** $F(x,y,z) = 4x^2+9y^2+3z^2, \quad x^2+y^2+z^2=1$ $\begin{cases} M = 9 \\ m = 3 \end{cases}$ at $\begin{matrix} \mathbf{e}_2 = [0\ 1\ 0]^T \\ \mathbf{e}_3 = [0\ 0\ 1]^T \end{matrix}$

$$3 = 3x^2+3y^2+3z^2 \le 4x^2+9y^2+3z^2 \le 9x^2+9y^2+9z^2 = 9$$

- **Example 6** $F(x,y,z) = \mathbf{x}^T A\mathbf{x} = 3x^2+3y^2+4z^2+4xy+2yz+2zx, \quad x^2+y^2+z^2 = \|\mathbf{x}\|^2 = 1$

$$= \mathbf{y}^T \Lambda \mathbf{y} = 6u^2+3v^2-w^2 \qquad u^2+v^2+w^2 = \|\mathbf{y}\|^2 = 1$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \begin{cases} \lambda_1 = 6 \\ \lambda_2 = 3 \\ \lambda_3 = -1 \end{cases} \quad Q = \begin{array}{c} \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt6} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt6} & \frac{-1}{\sqrt2} \\ \frac{1}{\sqrt3} & \frac{-2}{\sqrt6} & 0 \end{bmatrix} \\ \mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \end{array}, \quad \Lambda = \begin{bmatrix} 6 & & \\ & 3 & \\ & & -1 \end{bmatrix}$$

$$\begin{cases} M = 6 = \lambda_1 = \max_i \lambda_i \\ m = -3 = \lambda_3 = \min_i \lambda_i \end{cases} \text{ at } \begin{matrix} \mathbf{y} = \mathbf{e}_1, \quad \mathbf{x} = \mathbf{q}_1 \\ \mathbf{y} = \mathbf{e}_3, \quad \mathbf{x} = \mathbf{q}_3 \end{matrix}$$

(1) $Q=[q_1,\dots,q_n]$ orthogonal : (i) $\{q_1,\dots,q_n\}$ orthonormal $\Leftrightarrow Q^TQ=I_n$

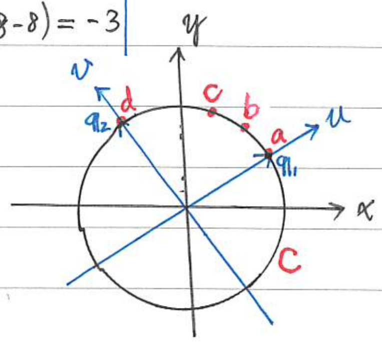
$$\Rightarrow \begin{cases} \text{(a) } \|Q\mathbf{x}\|=\|\mathbf{x}\| \quad (\text{保長}) \\ \text{(b) } Q^{-1}=Q^T \quad \Rightarrow Q:\text{ nonsingular} \Rightarrow Q:\text{ 1-1, onto } (\mathbb{R}^n\to\mathbb{R}^n) \\ \text{(c) } (Q^T)^TQ^T=QQ^T=I_n \Rightarrow Q^T:\ 〃 \quad \Rightarrow Q^T:\ 〃 \quad (\&\text{ 保長}) \\ \text{(d) } C=\{\mathbf{x}\in\mathbb{R}^n \mid \|\mathbf{x}\|=1\} \underset{Q^T}{\overset{Q}{\rightleftarrows}} C:\text{ 1-1 \& onto} \end{cases}$$

(2) $\mathbf{x}^TA\mathbf{x} = \mathbf{y}^TQ^TAQ\mathbf{y} = \mathbf{y}^T\Lambda\mathbf{y}$

$$\begin{cases} F(\mathbf{x}) = [x,y]\begin{bmatrix}1&2\\2&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \overset{\mathbf{x}=Q\mathbf{y}}{=\!=\!=} [u,v]\begin{bmatrix}2&0\\0&-3\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = F(\mathbf{y}) \\ F(x,y) = x^2+4xy-2y^2 =\!=\!= 2u^2-3v^2 = F(u,v) \end{cases}$$

	$\mathbf{y}=\begin{bmatrix}u\\v\end{bmatrix}$	$F(u,v)=2u^2-3v^2$	$\mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}$	$F(x,y)=x^2+4xy-2y^2$
(a)	$\begin{bmatrix}1\\0\end{bmatrix}$	2	$\frac{1}{\sqrt5}\begin{bmatrix}2\\1\end{bmatrix}$	$\frac15(4+8-2)=2$
(b)	$\frac15\begin{bmatrix}4\\3\end{bmatrix}$	$\frac{1}{25}(32-27)=\frac15$	$\frac{1}{5\sqrt5}\begin{bmatrix}5\\10\end{bmatrix}$	$\frac15(1+8-8)=\frac15$
(c)	$\frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac12(2-3)=-\frac12$	$\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$	$\frac1{10}(1+12-18)=-\frac12$
(d)	$\begin{bmatrix}0\\1\end{bmatrix}$	-3	$\frac{1}{\sqrt5}\begin{bmatrix}-1\\2\end{bmatrix}$	$\frac15(1-8-8)=-3$

$\mathbf{x}=Q\mathbf{y}=\frac{1}{\sqrt5}\begin{bmatrix}2&-1\\1&2\end{bmatrix}$



Principal Component Analysis (PCA) 主成分分析 (dimension reduction)

- data matrix $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$ $X_1 = X$, $X_2 = Y$

$$\begin{cases} \text{mean} : \mu_X = E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \\ \text{variance} : \sigma_X^2 = Var(X) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_X)^2 \xrightarrow[\text{去中心化}]{x_i \leftarrow x_i-\mu_X} \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{1}{n}XX^T = Cov(X,X) \\ \text{covariance} : \sigma_{XY} = Cov(X,Y) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_X)(y_i-\mu_Y) \overset{\prime\prime}{=} \frac{1}{n}\sum_{i=1}^{n} x_i y_i = \frac{1}{n}XY^T \end{cases}$$

- Covariance matrix: $C = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{YX} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{bmatrix} = \frac{1}{n}\begin{bmatrix} X \\ Y \end{bmatrix}[X^T\ Y^T] = \frac{1}{n}XX^T$

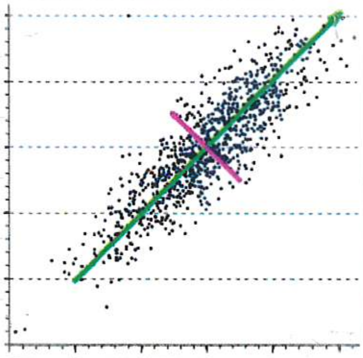
- $X_{d\times n}$ (去中心化), $C = \frac{1}{n}XX^T = \begin{bmatrix} Cov(X_1,X_1) & Cov(X_1,X_2) & Cov(X_1,X_d) \\ Cov(X_2,X_1) & Cov(X_2,X_2) & Cov(X_2,X_d) \\ \vdots & \vdots & \vdots \\ Cov(X_d,X_1) & Cov(X_d,X_2) & Cov(X_d,X_d) \end{bmatrix}_{d\times d}$

C: 对稱 $\Rightarrow$ (1) $\exists \{q_1,\ldots,q_d\}$ orthonormal, $Cq_i = \lambda_i q_i$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$

(2) $\begin{cases} F(u) = u^TCu, \ \|u\| = 1 & \max = \lambda_1 \text{ at } q_1 \\ \|u\|=1,\ u^Tq_1 = 0, & \max = \lambda_2 \ \text{〃}\ q_2 \\ \|u\| = 1,\ \begin{cases} u^Tq_1 = 0, \\ u^Tq_2 = 0, \end{cases} & \max = \lambda_3 \ \text{〃}\ q_3 \end{cases}$

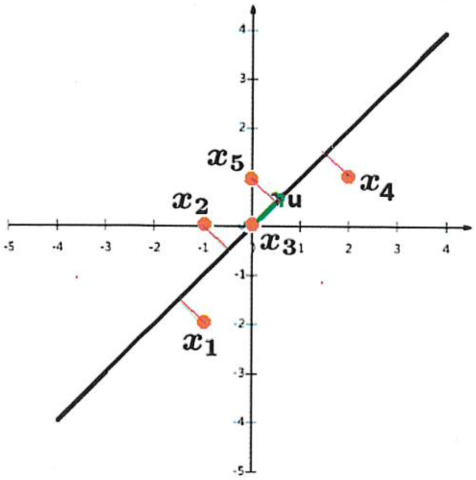
- Principal Component Analysis (PCA)

$$\max_{\|u\|=1} \frac{1}{n}\sum_{i=1}^{n}(x_i\cdot u)^2 = \max u^T\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i^T\right)u = \max u^T\left(\frac{1}{n}[x_1 \cdots x_n]\begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}\right)u$$
$$= \max u^T\left(\frac{1}{n}XX^T\right)u = \max_{\|u\|=1} u^TCu$$



- 定理 1 data matrix $X_{d\times n}$. (去中心化), $C = \frac{1}{n}XX^T = Q\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}Q^T$ $\quad \lambda_1 \geq \cdots \geq \lambda_d$

(1) $\begin{cases} \text{1st PC}: q_1, \lambda_1 \\ \text{2nd PC}: q_2, \lambda_2, \\ \vdots \\ \text{dth PC}: q_d, \lambda_d \end{cases}$ (2) $\begin{cases} \text{PCA後 data}: Y = Q^TX \\ C' = \frac{1}{n}YY^T = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{bmatrix} \end{cases}$



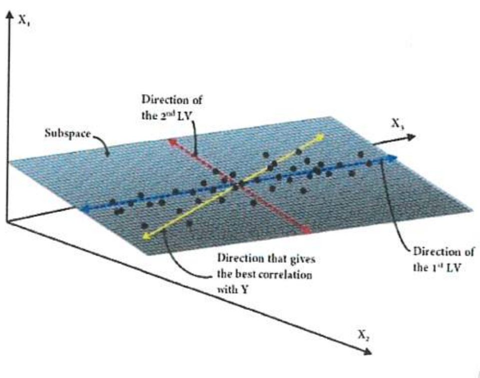
- Example

$$\begin{array}{l} LA: \\ ML: \end{array}\begin{bmatrix} 79 & 79 & 80 & 82 & 80 \\ 68 & 70 & 70 & 71 & 71 \end{bmatrix}\overset{\mu}{\begin{matrix} 80 \\ 70 \end{matrix}} \xrightarrow{\text{去中心化}} X = \begin{bmatrix} -1 & -1 & 0 & 2 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$C = \frac{1}{5}XX^T = \frac{1}{5}\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$, $\begin{cases} \lambda_1 = 2, & q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda_2 = \frac{2}{5}, & q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$

$\begin{cases} \text{1st PC}: q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \text{2nd PC}: q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$ $\quad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

PCA後 data: $\begin{cases} Y = Q^TX = \frac{1}{\sqrt{2}}\begin{bmatrix} -3 & -1 & 0 & 3 & 1 \\ -1 & 1 & 0 & -1 & 1 \end{bmatrix} & : \frac{5}{6}\ (83\%) \\ & : \frac{1}{6}\ (17\%) \\ C' = \frac{1}{5}YY^T = \begin{bmatrix} 2 & 0 \\ 0 & 2/5 \end{bmatrix} \end{cases}$



- Remarks

(1) $\{q_1, q_2, \ldots, q_d\}$ orthonormal

(2) $\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_d = \sigma_1^2 + \cdots + \sigma_d^2 = tr(C) \\ Y_1, Y_2, \ldots, Y_d \text{ 獨立} \end{cases}$

Singular value decomposition (SVD) 奇異值分解

No.
Date: / /

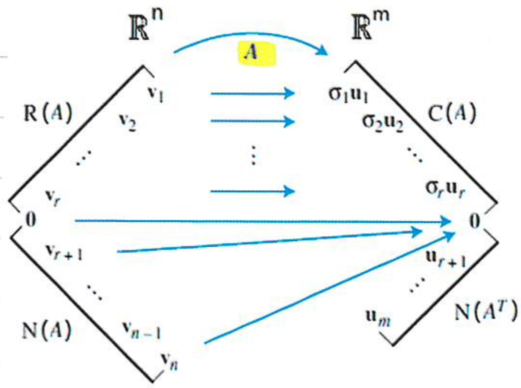
- $A_{m\times n}$, $\operatorname{rank}(A) = r$

  A^TA 对稱 $\Rightarrow \exists \{\mathbb{v}_1, \mathbb{v}_2, \dots, \mathbb{v}_r, \mathbb{v}_{r+1}, \dots, \mathbb{v}_n\}$ orthonormal eigenbasis

  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 = \dots = 0$ eigen values : $A^TA\mathbb{v}_i = \lambda_i \mathbb{v}_i$

  $\sqrt{\ }\downarrow$ $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0 = \dots = 0$ singular 〃 : $\sigma_i = \sqrt{\lambda_i}$

• 定理 1



(1) $\|A\mathbb{v}_i\|^2 = \lambda_i \geq 0$, 令 $A\mathbb{v}_i = \sigma_i \mathbb{u}_i$, $\|\mathbb{u}_i\| = 1$

(2) orthonormal bases ($\mathbb{R}^n$: $\mathcal{V} = \{\mathbb{v}_1, \dots, \mathbb{v}_n\}$)

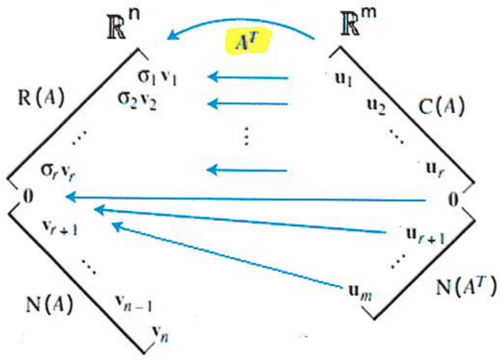
$$\begin{cases} R(A) : \{\mathbb{v}_1, \dots, \mathbb{v}_r\} & A\mathbb{v}_i = \sigma_i \mathbb{u}_i, & N(A) : \{\mathbb{v}_{r+1}, \dots, \mathbb{v}_n\} \\ C(A) : \{\mathbb{u}_1, \dots, \mathbb{u}_r\} & A^T\mathbb{u}_i = \sigma_i \mathbb{v}_i, & N(A^T) : \{\mathbb{u}_{r+1}, \dots, \mathbb{u}_m\} \end{cases}$$

($\mathbb{R}^m$: $\mathcal{U} = \{\mathbb{u}_1, \dots, \mathbb{u}_m\}$)

証

(1) $A\mathbb{v}_i \cdot A\mathbb{v}_j = \mathbb{v}_i \cdot A^TA\mathbb{v}_j = \lambda_j \mathbb{v}_i \cdot \mathbb{v}_j = \begin{cases} \lambda_i & i = j \\ 0 & i \neq j \end{cases}$

(2) $\begin{cases} \{\mathbb{v}_1, \dots, \mathbb{v}_r\}\ R(A) \text{基底} \Rightarrow \{A\mathbb{v}_1, \dots, A\mathbb{v}_r\}\ C(A) \text{基底} \\ A^T\mathbb{u}_i = A^T\left(\frac{1}{\sigma_i}A\mathbb{v}_i\right) = \frac{1}{\sigma_i}A^TA\mathbb{v}_i = \sigma_i \mathbb{v}_i \end{cases}$



$A^T = V\Sigma^T U^T$

• 定理 2 (SVD)

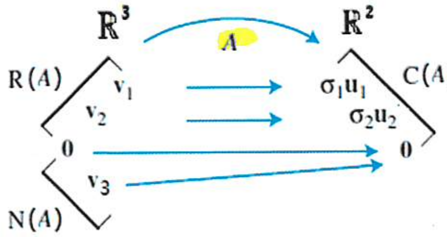
$$A_{m\times n} = [\mathbb{u}_1 \cdots \mathbb{u}_r | \mathbb{u}_{r+1} \cdots \mathbb{u}_m] \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} \mathbb{v}_1^T \\ \vdots \\ \mathbb{v}_r^T \\ \vdots \\ \mathbb{v}_n^T \end{bmatrix}$$

$$= U_{m\times m} \Sigma_{m\times n} V^T_{n\times n} \qquad \begin{cases} U^TU = I_m \\ V^TV = I_n \end{cases}$$

$$= \underbrace{\underbrace{\underbrace{\sigma_1 \mathbb{u}_1 \mathbb{v}_1^T}_{\text{rank}=1} + \sigma_2 \mathbb{u}_2 \mathbb{v}_2^T}_{\text{rank}=2} + \dots + \sigma_r \mathbb{u}_r \mathbb{v}_r^T}_{\text{rank}=r}$$

証: $AV = [A\mathbb{v}_1 \cdots A\mathbb{v}_r | \mathbb{0} \cdots \mathbb{0}] = [\sigma_1 \mathbb{u}_1 \cdots \sigma_r \mathbb{u}_r | \mathbb{0} \cdots \mathbb{0}] = U\Sigma$

• 定理 3 (Simplified SVD)



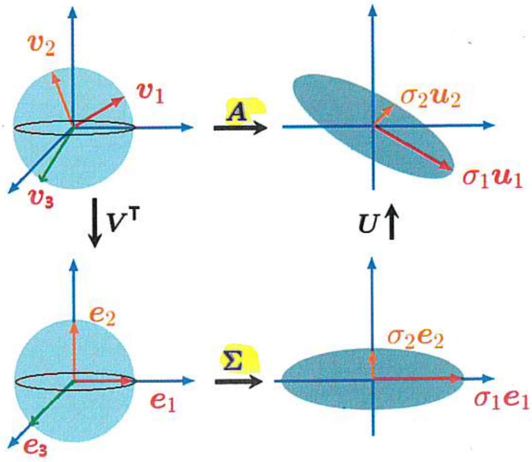
$$A_{m\times n} = [U_r | U_{n-r}] \left[\begin{array}{c|c} D_r & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} V_r^T \\ \hline V_{n-r}^T \end{bmatrix}$$

$$= U_r D_r V_r^T$$

• Example

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$$\begin{cases} \lambda_1 = 360, & \mathbb{v}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, & A\mathbb{v}_1 = \begin{bmatrix} 18 \\ 6 \end{bmatrix}, & \sigma_1 = 6\sqrt{10}, & \mathbb{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \lambda_2 = 90, & \mathbb{v}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, & A\mathbb{v}_2 = \begin{bmatrix} 3 \\ -9 \end{bmatrix}, & \sigma_2 = 3\sqrt{10}, & \mathbb{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ \lambda_3 = 0, & \mathbb{v}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, & A\mathbb{v}_3 = \mathbb{0}, & & \end{cases}$$



$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \left[\begin{array}{cc|c} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{array}\right] \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6\sqrt{10} \\ 6\sqrt{10} \end{bmatrix}_{\mathcal{U}} = \begin{bmatrix} 24 \\ -12 \end{bmatrix} \leftarrow \begin{bmatrix} 6\sqrt{10}\cdot 1 \\ 3\sqrt{10}\cdot 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \leftarrow \frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{V}}$$

- **Remarks:** (1) $A_{n\times n}$: nonsingular $\Leftrightarrow$ n non-zero singular values

(2) $\begin{cases} A^TA\, v_i = \lambda_i v_i, & 1\le i\le n \\ AA^T\, u_i = \lambda_i u_i, & 1\le i\le m \end{cases}$ (或 $A^TA = V\left[\begin{array}{c|c} D_r^2 & 0 \\ \hline 0 & 0 \end{array}\right]V^T$, $D_r^2=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_r\end{bmatrix}$)

(〃 $AA^T = U\left[\begin{array}{c|c} D_r^2 & 0 \\ \hline 0 & 0 \end{array}\right]U^T$, 〃)

(PCA: $A = X$)

- **Image compression**

$$A = \underbrace{\underbrace{\sigma_1 u_1 v_1^T}_{1} + \underbrace{\sigma_2 u_2 v_2^T}_{1} + \cdots + \underbrace{\sigma_k u_k v_k^T}_{1}}_{\text{rank} = k} + \cdots + \underbrace{\sigma_r u_r v_r^T}_{1}$$

rank: 1 1 1 1

$$\text{保留影像比例} = \frac{\sigma_1+\sigma_2+\cdots+\sigma_k}{\sigma_1+\sigma_2+\cdots+\sigma_r}$$

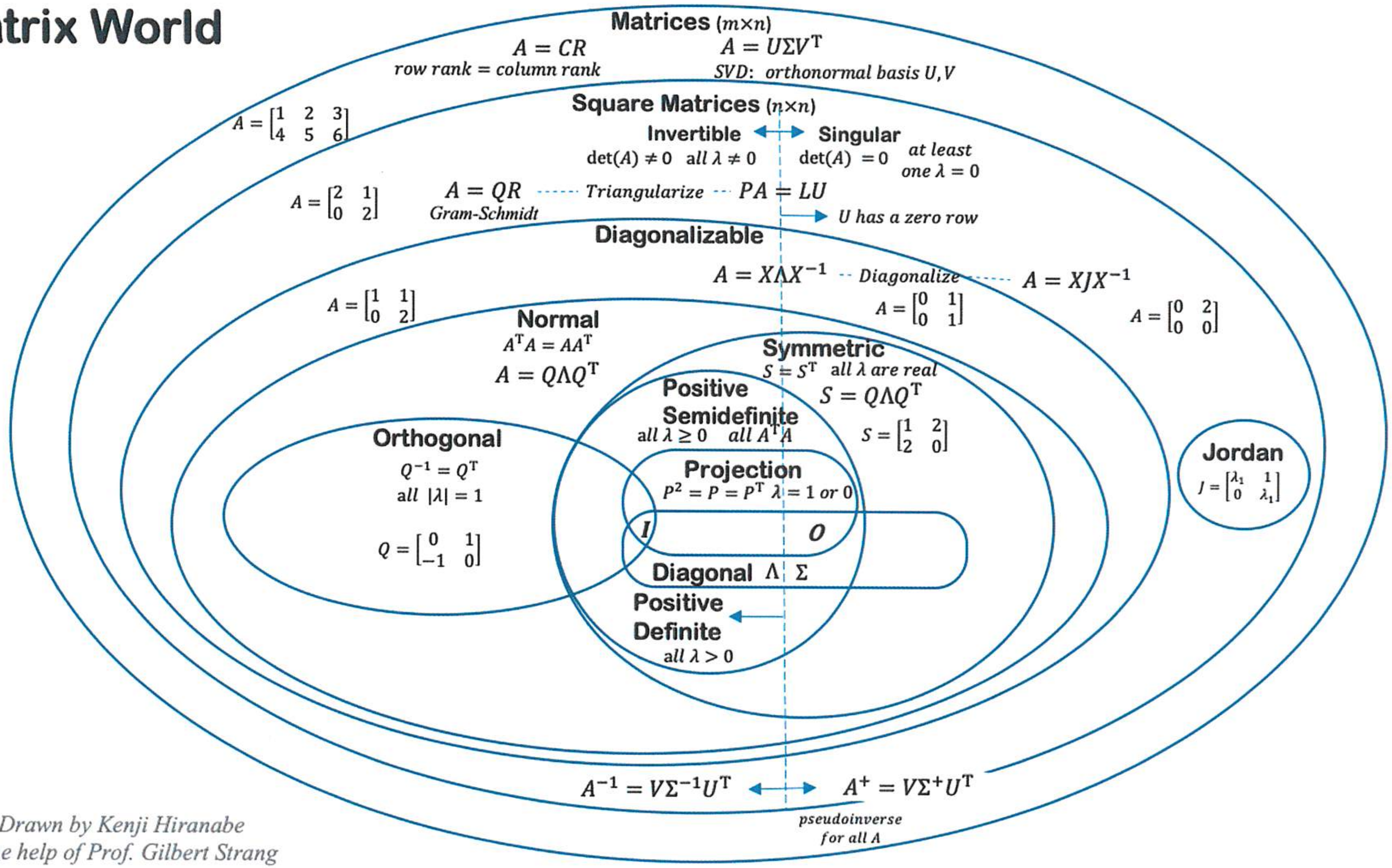
SVD 壓縮後的效果

原圖 80%

原圖 50%

70% 90%

Matrix World



(v1.3) Drawn by Kenji Hiranabe
with the help of Prof. Gilbert Strang